Probability Theory on Product Spaces

Michael Betancourt

August 2026

All probability theory chapters can be found here.

A repository containing all of the files used to generate this chapter is available on GitHub.

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Too this point, our discussion of conditional probability theory has been a bit hampered by its abstraction. Fortunately, both the theory and application of conditional probability theory become much more straightforward when applied to the *product spaces* that dominate practice. In particular, this application allows us to build up sophisticated probability distributions from simpler, more manageable components. Not unlike the popular Danish toy: Bilofix.

In this chapter, we will work through the application of conditional probability theory to product spaces step by step, with a specific emphasis on the probability density functions and directed graphical model representations that are particularly useful in practical applications.

1 Product Probability Spaces

Before considering conditional probability theory, let's begin by discussing the probabilistic objects that we can derive from the structure of an arbitrary product space.

1.1 Product Space Review

We discussed product spaces in depth in Chapter 3. To facilitate the discussion here, however, let's briefly review the basics.

Given a finite collection of *component spaces*,

$$\{X_1, \ldots, X_I\},$$

we can construct the *product space*,

$$X^{1:I} = \times_{i \in 1:I}^{I} X_i = \times_{i=1}^{I} X_i,$$

where $1:I$ is a *multi-index* variable that denotes the indices of all of the component spaces,

$$1:I = (1, \ldots, I).$$

Each point in the product space $X^{1:I}$ is uniquely determined by a point in each of the component spaces. In particular, any variable taking values in the product space,

$$x_{1:I} \in X^{1:I},$$

can be decomposed into an n-tuple

$$x_{1:I} = (x_1, \ldots, x_I) \in X^{1:I}$$

of component variables taking values in each component space,

$$x_i \in X_i.$$

Any subset of component indices,

$$\mathsf{i} = (i_1, \ldots, i_J) \subset 1:I$$

defines its own product space,

$$X^{\mathsf{i}} = \times_{i \in \mathsf{i}} X_i,$$

where every point is specified by a point in the included component spaces,

$$x_{\mathsf{i}} = (x_{i_1}, \ldots, x_{i_J}) \in X^{\mathsf{i}}$$

with

$$x_{i_j} \in X_{i_j}.$$

A particular point

$$\tilde{x}_{\mathsf{i}} \in X^{\mathsf{i}}$$

does not completely specify a point in the full product space, but it does constrain one. The remaining degrees of freedom define a subspace of the full product space,

$$X^{\mathsf{i}^c} \times \{\tilde{x}_{\mathsf{i}}\} \subset X^{1:I}$$

where

$$\mathsf{i}^c = 1:I \quad \mathsf{i} = \{i \mid i \in 1:I \text{ and } i \notin \mathsf{i}\}.$$

These subspaces are known as *cross sections*.

Every point in a cross section

$$(x_{\mathsf{i}^c})_{\tilde{x}_{\mathsf{i}}} \in X^{\mathsf{i}^c} \times \{\tilde{x}_{\mathsf{i}}\},$$

is uniquely determined by points in the complementary component spaces,

$$x_i \in X_i$$

for

$$i \in \mathsf{i}^c.$$

At the same time, these component points uniquely determine a point in the complementary product space

$$X^{\mathsf{i}^c} = \times_{i \in \mathsf{i}^c} X_i.$$

Consequently, each cross section can be treated as a copy of the complementary product space,

$$X^{\mathsf{i}^c} \times \{\tilde{x}_{\mathsf{i}}\} \cong X^{\mathsf{i}^c}.$$

Altogether, a choice of component indices i *decomposes* the full product space into a smaller product space and a collection of cross sections. By specifying a point in the smaller produce space,

$$x_{\mathsf{i}} \in X^{\mathsf{i}},$$

and a point in the corresponding cross section,

$$(x_{\mathsf{i}^c})_{x_{\mathsf{i}}} \in X^{\mathsf{i}^c} \times \{x_{\mathsf{i}}\},$$

we completely determine a point in the full product space,

$$((x_{\mathsf{i}^c})_{x_{\mathsf{i}}}, x_{\mathsf{i}}) \in X^{1:I}.$$

This decomposition is neatly organized by the corresponding *projection function*,

$$\varpi_{\mathsf{i}} : X^{1:I} \to X^{\mathsf{i}}.$$

This projection function outputs the smaller product space, while its level sets are exactly the cross sections,

$$\varpi_{\mathsf{i}}^{-1}(\tilde{x}_{\mathsf{i}}) = X^{\mathsf{i}^c} \times \{\tilde{x}_{\mathsf{i}}\}.$$

1.2 Product σ-Algebras

Just as we can build up each point in a product space from points in the component spaces, we can often build up structures over a product space from structures over the component spaces. This includes many probabilistic structures.

Let's say, for example, that each component space is equipped with its own component σ-algebra $\mathcal{X}_i$, each of which defines measurable component subsets. A rectangular subset build up from measurable component subsets,

$$\mathsf{x} = \times_{i \in 1:I} \mathsf{x}_i$$

with

$$\mathsf{x}_i \in \mathcal{X}_i,$$

is compatible with *all* of the component-wise structures at the same time. Consequently, it is a natural candidate for a measurable subset over the full product space.

The only problem with these candidates is they do not form a consistent σ-algebra. Specifically, their unions and intersections define non-rectangular subsets that aren't neatly related to the component σ-algebras. The solution is to just include all of these successors, *generating* a **product σ-algebra**, $\mathcal{X}^{1:I}$, from the rectangular subsets.

One useful feature of product σ-algebras is that they are always compatible with all of the projection functions that we can define over a product space. More formally, given the component indices i, the projection function ϖ_{i} is always $(\mathcal{X}^{1:I}, \mathcal{X}^{\mathsf{i}})$-measurable. Because of this, we rarely have to worry about measurability on finite product spaces.

1.3 Independent Product Measures and Probability Distributions

Product σ-algebras are not the only probabilistic structure that we can derive from component probabilistic structure. We can also derive entire measures and probability distributions.

Consider each component space being equipped with not only a component σ-algebra $\mathcal{X}_i$, but also a component measure

$$\mu_i : \mathcal{X}_i \to [0, \infty].$$

Each of these component measures defines allocations to component measurable subsets. In turn, these component allocations define a notion of allocation to measurable rectangular subsets.

Specifically, given the measurable rectangular subset,

$$\mathsf{x} = \times_{i \in 1:I} \mathsf{x}_i$$

with

$$\mathsf{x}_i \in \mathcal{X}_i,$$

we can define the product allocations

$$\begin{aligned}\mu^{1:I}(\mathsf{x}) &= \mu^{1:I}\left(\times_{i \in 1:I} \mathsf{x}_i\right) \\ &\equiv \prod_{i=1}^{I} \mu_i(\mathsf{i}).\end{aligned}$$

To define the allocations for arbitrary measurable product subsets, we have to appeal once again to Carathéodory's extension theorem. By construction, every subset in a product σ-algebra can be decomposed into a countable union of rectangular subsets,

$$\mathsf{x} = \bigcup_j \times_{i \in 1:I} \mathsf{x}_{ji}.$$

Appealing to countable additivity, we can then define the measure allocated to an arbitrary measurable product subset as

$$\begin{aligned}\mu^{1:I}(\mathsf{x}) &= \mu^{1:I}\left(\bigcup_j \times_{i \in 1:I} \mathsf{x}_{ji}\right) \\ &= \sum_j \mu^{1:I}\left(\times_{i \in 1:I} \mathsf{x}_{ji}\right) \\ &= \sum_j \prod_{i=1}^{I} \mu_i\left(\mathsf{x}_{ji}\right).\end{aligned}$$

When each component measure is σ-finite, these derived allocations define a unique product measure

$$\mu^{1:I} : \mathcal{X}^{1:I} \to [0, \infty].$$

Because each component measure acts independently of the others, this derived measure is often referred to as an **independent product measure**.

If each component measure is not only σ-finite but also a probability distribution, then the derived measure will satisfy all of the Kolmogorov axioms. Consequently, component probability distributions define an **independent product probability distribution** over the corresponding product space. Given how cumbersome this name is, however, shorthands like **independent product distribution** are common.

One nice feature of this construction is that it does not lose any information. In particular, we can always recover any of the component measures by pushing an independent product measure forward along the corresponding projection function,

$$\mu_i = (\varpi_i)_* \mu.$$

We've already encountered an independent product measure in Chapter 4: multivariate real spaces $\mathbb{R}^I$ are built up from I component real lines. Fixing the metric on each of these real lines defines σ-finite component Lebesgue measures, and the independent product of these component Lebesgue measures formally defines the multivariate Lebesgue measure.

1.4 The Fubini-Tonelli Theorem

One of most useful properties of independent product measures is that their integrals can be computed entirely from component-wise integrals.

Before going into more detail, however, let's briefly discuss notation. The integral of a sufficiently well-behaved function

$$g : X^{1:I} \to \mathbb{R}.$$

with respect to an independent product measure is most compactly denoted

$$\mathbb{I}_{\mu^{1:I}}[g] = \int \mu^{1:I}(\mathrm{d}x_{1:I})\, g(x_{1:I})$$

Often, however, it often more useful to list the components out explicitly using the integral notation

$$\mathbb{I}_{\mu^{1:I}}[g] = \int \mu^{1:I}(\mathrm{d}x_1, \ldots, \mathrm{d}x_I)\, g(x_1, \ldots, x_I).$$

This is especially useful when we denote the component spaces not with indices, but rather different variables names, such as

$$\mathbb{I}_{\mu^{1:3}}[g] = \int \mu^{1:3}(\mathrm{d}x, \mathrm{d}y, \mathrm{d}z)\, g(x, y, z).$$

With notation settled, let's now consider two measurable component spaces X_1 and X_2 that are equipped with the σ-finite measures μ_1 and μ_2, respectively. The **Fubini-Tonelli theorem** (Folland 1999) shows that the integral of an integrable, measurable function

$$g : X_1 \times X_2 \to \mathbb{R}$$

with respect to the independent product measure,

$$\mathbb{I}_{\mu^{1,2}}[g] = \int \mu^{1,2}(\mathrm{d}x_1, \mathrm{d}x_2)\, g(x_1, x_2),$$

can be computed by *nesting* or *iterating* component integrals in either order,

$$\begin{aligned}
\mathbb{I}_{\mu^{1,2}}[g] &= \int \mu_1(\mathrm{d}x_1) \left[\int \mu_2(\mathrm{d}x_2)\, g(x_1, x_2) \right] \\
&= \int \mu_2(\mathrm{d}x_2) \left[\int \mu_1(\mathrm{d}x_1)\, g(x_1, x_2) \right].
\end{aligned}$$

By repeatedly applying this result, we can decompose integrals with respect to independent product measures on arbitrary product spaces into any sequence of component-wise integrals. More formally, for any permutation of the component indices,

$$(s(1), \ldots, s(I))$$

we can compute the independent product integral with the nested component integrals

$$\mathbb{I}_{\mu^{1:I}}[g] = \int \mu_{s(1)}(\mathrm{d}x_{s(1)}) \, ... \int \mu_{s(I)}(\mathrm{d}x_{s(I)}) \, g(x_1, ..., x_I).$$

To make this a bit more concrete, consider the product space $\mathbb{Z}^I$ built up from I integer spaces, each of which is naturally equipped with a counting measure χ_i. The independent product of these component counting measures, $\chi^{1:I}$ defines a product measure over $\mathbb{Z}^I$.

Because each χ_i is σ-finite, integration with respect to $\chi^{1:I}$ can be implemented by nested component integrals. Moreover, in this case the component integrals can be evaluated directly by summation.

Consequently, $\chi^{1:I}$ integrals can be evaluated by any sequence of nested summations, such as

$$\begin{aligned}
&\int \chi^{1:3}(\mathrm{d}x_1, \mathrm{d}x_2, \mathrm{d}x_3) \, g(x_1, x_2, x_3) \\
&\qquad = \sum_{x_1} \sum_{x_2} \sum_{x_3} g(x_1, x_2, x_3) \\
&\qquad = \sum_{x_1} \left[\sum_{x_2} \left[\sum_{x_3} g(x_1, x_2, x_3) \right] \right].
\end{aligned}$$

Another common example is a multivariate Lebesgue measure over $\mathbb{R}^I$. Because the component Lebesgue measures λ_i are all σ-finite, we can implement multivariate Lebesgue integrals by nesting one-dimensional Lebesgue integrals. When the integrand is sufficiently well-behaved, we can then evaluate these component Lebesgue integrals as Riemann integrals.

For instance, the integral of the integrand

$$f : \mathbb{R}^3 \to \mathbb{R}$$

can be calculated with any sequence of nested Riemann integrals, such as

$$\begin{aligned}
&\int \lambda^{1:3}(\mathrm{d}x_1, \mathrm{d}x_2, \mathrm{d}x_3) \, g(x_1, x_2, x_3) \\
&\qquad = \int \lambda_2(\mathrm{d}x_2) \left[\int \lambda_1(\mathrm{d}x_1) \left[\int \lambda_3(\mathrm{d}x_3) \, g(x_1, x_2, x_3) \right] \right] \\
&\qquad = \int \mathrm{d}x_2 \int \mathrm{d}x_1 \int \mathrm{d}x_3 \, g(x_1, x_2, x_3)
\end{aligned}$$

or

$$\begin{aligned}
&\int \lambda^{1:3}(\mathrm{d}x_1, \mathrm{d}x_2, \mathrm{d}x_3) \, g(x_1, x_2, x_3) \\
&\qquad = \int \lambda_3(\mathrm{d}x_3) \left[\int \lambda_2(\mathrm{d}x_2) \left[\int \lambda_1(\mathrm{d}x_1) \, g(x_1, x_2, x_3) \right] \right] \\
&\qquad = \int \mathrm{d}x_3 \int \mathrm{d}x_2 \int \mathrm{d}x_1 \, g(x_1, x_2, x_3).
\end{aligned}$$

Importantly, the Fubini-Tonelli theorem does not require that the component spaces are homogeneous. Consider, for example, the product of a rigid real line and an integer space,

$$X = \mathbb{R} \times \mathbb{Z}.$$

The two component spaces are naturally equipped with a Lebesgue measure and counting measure, respectively. The independent product of these two component measures defines a kind of mixed measure over X.

Integrals of sufficiently well-behaved functions with respect to this mixed measure can implemented by iterating summation and Riemann integration,

$$\begin{aligned}
\mathbb{I}_{\mu^{1,2}}[g] &= \int \mu_1(\mathrm{d}x_1) \left[\int \mu_2(\mathrm{d}x_2)\, g(x_1, x_2) \right] \\
&= \int \lambda(\mathrm{d}x_1) \left[\int \chi(\mathrm{d}x_2)\, g(x_1, x_2) \right] \\
&= \int \mathrm{d}x_1 \left[\sum_{x_2} g(x_1, x_2) \right]
\end{aligned}$$

or, equivalently,

$$\begin{aligned}
\mathbb{I}_{\mu^{1,2}}[g] &= \int \mu_2(\mathrm{d}x_2) \left[\int \mu_1(\mathrm{d}x_1)\, g(x_1, x_2) \right] \\
&= \int \chi(\mathrm{d}x_1) \left[\int \lambda(\mathrm{d}x_2)\, g(x_1, x_2) \right] \\
&= \sum_{x_1} \left[\int \mathrm{d}x_2\, g(x_1, x_2) \right].
\end{aligned}$$

1.5 General Product Measures and Probability Distributions

Once we have defined a product σ-algebra, we are not limited to only independent product measures. We can always define more general product measures axiomatically. Specifically, any countably-addictive function

$$\nu : \mathcal{X}^{1:I} \to [0, \infty]$$

defines a measure over the product space $X^{1:I}$, while while any countably-additive function

$$\pi : \mathcal{X}^{1:I} \to [0, 1].$$

defines a probability distribution.

Once we introduce general product measures, however, we have to be weary of the potential for terminological confusion. Here, I have used "product measure" to refer to any measure over a product space. In particular, "product" refers to the structure of the *space* and not the

structure of the *measure*. If one were to assume the latter, however, then "product measure" would more naturally refer to the specific independent product measures that we derived in the previous section.

One way to avoid this potential confusion is to be a bit more descriptive, referring to arbitrary product measures as **joint product measures**, or simply **joint measures**. The adjective "joint" here implies that the measure can't necessarily be decomposed into separate component contributions; it also nicely complements the use of "independent" for those exceptional measures that can be decomposed into separate component contributions.

Equivalently, I will refer to arbitrary product probability distributions as **joint product probability distributions**, **joint probability distributions**, or even just **joint distributions** for short.

Let's pause briefly to review. Given a finite collection of measurable spaces,

$$\{(X_1, \mathcal{X}_1), ..., (X_I, \mathcal{X}_I)\},$$

we can immediately construct a product space $X^{1:I}$ and a product σ-algebra, $\mathcal{X}^{1:I}$. In other words, component measurable spaces define a measurable product space.

Once we've constructed the measurable product space $(X^{1:I}, \mathcal{X}^{1:I})$, we can then define arbitrary joint measures and probability distributions. At the same time, a finite collection of measure spaces

$$\{(X_1, \mathcal{X}_1, \mu_i), ..., (X_I, \mathcal{X}_I, \mu_i)\},$$

defines not only a product space and product σ-algebra, but also an independent product measure. Component measure spaces also define a measure product space.

In addition to *deriving* independent product measures from explicit component measures, we can also define them axiomatically as special joint measures. From this more abstract perspective, an independent product measure is any measure that satisfies

$$\mu(\mathsf{x}) = \sum_{i \in 1:I} (\varpi_i)_* \mu(\mathsf{x}_i)$$

for all rectangular subsets

$$\mathsf{x} = \times_{i \in 1:I} \mathsf{x}_i$$

with measurable components,

$$\mathsf{x}_i \in \mathcal{X}_i.$$

1.6 Component Means

If a component space can be embedded into a rigid real line, then the corresponding projection function defines a function that can be integrated. Formally, given a projection function,

$$\varpi_i : X^{1:I} \to X_i,$$

an embedding function,

$$\iota : X_i \to \mathbb{R},$$

and a joint probability distribution,

$$\pi : \mathcal{X}^{1:I} \to [0, 1],$$

we can construct the expectation value

$$\mathbb{E}_{\pi}[\iota \circ \varpi_i] \,.$$

Using the transformation properties of expectation values that we discussed in Chapter 7, we can also write this expectation value as an expectation value on the corresponding component space,

$$\mathbb{E}_{\pi}[\iota \circ \varpi_i] = \mathbb{E}_{\pi}[(\varpi_i)^* \iota] = \mathbb{E}_{(\varpi_i)_*}[\iota] \,.$$

From this component-wise perspective, the expectation value is just the mean of the pushforward distribution!

When taking the embedding for granted,

$$\mathbb{E}_{\pi}[\varpi_i] \equiv \mathbb{E}_{\pi}[\iota \circ \varpi_i] \,,$$

we often refer to the expectation of a projection function as a **component mean**. From this component mean, we can then build up more general component moments and cumulants accordingly.

The convention for denoting a component mean varies wildly across different communities. For example, it's not uncommon to encounter a notation like

$$\mathbb{E}_{\pi}[x_i],$$

where the projection function is completely implied by the component variable.

1.7 Expanding Product σ-Algebras

We'll end this section on a technical note that can be safely skipped by those not interested in the technical notes.

Product σ-algebras inherit many of the nice features that are shared by all of the component σ-algebras. For example, if the component σ-algebras are all Hausdorff, then the product σ-algebra will also be Hausdorff.

That said, not all useful properties are preserved by this construction. For instance, even if all of the component σ-algebras include all of their subsets, the product σ-algebra might not. This inconsistency obstructs the construction of complete product measures even when the component measures are themselves complete.

To facilitate the construction of complete product measures, we have to expand the initial product σ-algebra. Specifically, we have to include rectangular subsets where only *some* of the components are measurable, as well as the subsets that we can derive from their unions and intersections.

This raises the question of how we should define the action of an independent product measure on one of these rectangular subsets of mixed measurability. Typically we just set the allocations to zero, so that the new subsets are all null subsets.

When the additional subsets in the expanded product σ-algebra are all null subsets, they have have no impact on practical applications. The difference between a nominal product σ-algebra and an expanded one can safely be ignored. The distinction, however, is important in more theoretical calculations. Consequently, readers moving on to more technical treatments of probability theory should prepare themselves for this subtlety.

2 Conditional Probability Theory on Product Spaces

At this point, we have discussed the explicit construction of independent product measures, but only the implicit existence of more general, joint product measures. How do we construct explicit joint product measures? With our good friend, conditional probability theory.

Applying conditional probability theory to the projection functions on a product space allows us to build up joint product measures from more manageable probabilistic pieces.

2.1 Conditioning on Projection Functions

Recall that any subset of components,

$$\mathsf{i} \subset 1 : I,$$

defines a projection function that maps the full product space into a smaller product subspace,

$$\varpi_{\mathsf{i}} : X^{1:I} \to X^{\mathsf{i}}.$$

If each component space X_i is equipped with a component σ-algebra, then we can construct a product σ-algebra over both $X^{1:I}$ and X^{i}.

If the component σ-algebras are all Hausdorff, then these product σ-algebra will also be Hausdorff. This allows us to also define a subspace σ-algebra over the cross sections of the projection function,

$$\varpi_{\mathsf{i}}^{-1}(x_{\mathsf{i}}) = X^{\mathsf{i}^c} \times \{x_{\mathsf{i}}\}.$$

In fact, the subspace σ-algebra over these cross sections are all equivalent to the complementary product σ-algebra $\mathcal{X}^{\mathsf{i}^c}$!

Because the projection function ϖ_{i} is measurable with respect to all of these σ-algebras, it disintegrates any joint probability distribution into a marginal probability distribution over X^{i},

$$(\varpi_{\mathsf{i}})_{*}\pi(\mathsf{x}_{\mathsf{i}}),$$

and a conditional probability kernel,

$$\pi^{\varpi_{\mathsf{i}}}(\mathsf{x} \mid x_{\mathsf{i}}),$$

consisting of conditional probability distributions that concentrate on each cross section. If we interpret the conditional probability distributions as not just concentrating but actually being defined on these cross sections, then we can also write the kernel in terms of cross section variables,

$$\pi^{\varpi_{\mathsf{i}}}((\mathsf{x}_{\mathsf{i}^c})_{x_{\mathsf{i}}} \mid x_{\mathsf{i}}).$$

This formal notation takes nothing for granted, but it's also sufficiently dense that it can be difficult to parse. If we're conditioning with only projection functions, however, then some of this notation is redundant and amenable to streamlining.

In particular, the variable to the right of the conditioning bar completely determines both the relevant projection function and the corresponding cross section. Consequently, in this case we can safely overload our notation a bit, writing

$$(\varpi_{\mathsf{i}})_{*}\pi(\mathsf{x}_{\mathsf{i}}) = \pi(\mathsf{x}_{\mathsf{i}})$$

and

$$\pi^{\varpi_{\mathsf{i}}}((\mathsf{x}_{\mathsf{i}^c})_{x_{\mathsf{i}}} \mid x_{\mathsf{i}}) = \pi(\mathsf{x}_{\mathsf{i}^c} \mid x_{\mathsf{i}}).$$

The arguments fully differentiate each probabilistic object without any risk of ambiguity. At least, that is, under the assumption that we are conditioning with only projection functions. While these particular disintegrations dominate practical applications, they are not completely universal. Consequently, we do have to be on the lookout for the occasional exception.

When working with only a few component spaces, this streamlined notation becomes even easier to parse in integral notation. For instance, given four component spaces

$$(X_1, X_2, X_3, X_4),$$

the conditional expectations defined by the disintegration of a joint measure with respect to the projection function

$$\varpi_{(2,4)} : X^{(1,2,3,4)} \to X^{(2,4)}$$

can be written as

$$\int \pi(\mathrm{d}x_1, \mathrm{d}x_3 \mid x_2, x_4)\, g(x_1, x_3).$$

Similarly, the law of total expectation can be written as

$$\begin{aligned}\int \pi(\mathrm{d}x_1,&\mathrm{d}x_2, \mathrm{d}x_3, \mathrm{d}x_4)\, g(x_1, x_2, x_3, x_4) \\ &= \int \pi(\mathrm{d}x_2, \mathrm{d}x_4) \int \pi(\mathrm{d}x_1, \mathrm{d}x_3 \mid x_2, x_4)\, g(x_1, x_2, x_3, x_4),\end{aligned}$$

or even just

$$\pi(\mathrm{d}x_1, \mathrm{d}x_2, \mathrm{d}x_3, \mathrm{d}x_4) = \pi(\mathrm{d}x_2, \mathrm{d}x_4)\, \pi(\mathrm{d}x_1, \mathrm{d}x_3 \mid x_2, x_4).$$

2.2 Conditional Decomposition of Joint Probability Distributions

The utility of conditional probability theory is not just limited to a single application. After disintegrating a joint probability distribution into a marginal distribution and conditional probability kernel, we can often disintegrate the marginal distribution into even simpler pieces.

Given a subset of component indices,

$$\mathsf{i} \subset 1 : I$$

the projection function ϖ_{i} disintegrates a joint probability distribution π into a marginal distribution a conditional probability kernel. At this point, a smaller subset of component indices

$$\mathsf{i}' \subset \mathsf{i} \subset 1 : I$$

defines a new projection function

$$\varpi_{\mathsf{i}'} : X^{\mathsf{i}} \to X^{\mathsf{i}'}$$

that disintegrates $(\varpi_{\mathsf{i}})_* \pi$ into a new conditional probability kernel

$$\pi(\mathsf{x}_{\mathsf{i}\,\mathsf{i}'} \mid x'_{\mathsf{i}})$$

defined over the cross sections

$$X^{\mathsf{i}\,\mathsf{i}'} \times \{x'_{\mathsf{i}}\},$$

, and a new marginal distribution

$$\pi(\mathsf{x}_{\mathsf{i}'})$$

defined over the smaller product space

$$X^{\mathsf{i}'}.$$

Joint expectation values can be evaluated by nesting the two conditional expectations and one marginal expectation,

$$\begin{aligned}\int \pi(\mathrm{d}x_{1:I})\, &g(x_{\mathsf{i}^{\mathrm{c}}}, x_{\mathsf{i}\,\mathsf{i}'}, x_{\mathsf{i}'}) \\ &= \int \pi(\mathsf{x}_{\mathsf{i}'}) \int \pi(\mathsf{x}_{\mathsf{i}\,\mathsf{i}'} \mid x_{\mathsf{i}'}) \int \pi(\mathsf{x}_{\mathsf{i}^{\mathrm{c}}} \mid x_{\mathsf{i}\,\mathsf{i}'})\, g(x_{\mathsf{i}^{\mathrm{c}}}, x_{\mathsf{i}\,\mathsf{i}'}, x_{\mathsf{i}'}).\end{aligned}$$

Symbolically, we often write this as

$$\pi(\mathrm{d}x_{1:I}) = \pi(\mathsf{x}_{\mathsf{i}'}) \, \pi(\mathsf{x}_{\mathsf{i}\,\mathsf{i}'} \mid x_{\mathsf{i}'}) \, \pi(\mathsf{x}_{\mathsf{i}^c} \mid x_{\mathsf{i}\,\mathsf{i}'}).$$

If i' contains more than one component, then we can repeat this process, disintegrating the new marginal distribution at each step until we are satisfied or left with a single component space. One of more difficult aspects of trying to implement a recursive decomposition like this in practice is just keeping track of all of the intermediate spaces.

Conveniently, we can we systematize these recursive decompositions by organizing each step into a subset of active component indices. Because each component is active at least once but cannot be active more than once, these subsets form a partition of

$$1 : I = \{1, \ldots, I\}.$$

More formally, a decomposition with K steps is defined by K index subsets

$$\mathsf{i}_k \subset \{1, \ldots, I\}$$

which are non-empty,

$$\mathsf{i}_k \neq \emptyset,$$

mutually disjoint,

$$\mathsf{i}_k \cap \mathsf{i}_{k'} = \emptyset,$$

and include all of the component indices,

$$\bigcup_k \mathsf{i}_k = \{1, \ldots, I\}.$$

These subsets define a sequence of product spaces

$$\times i \in \cup_{k'=k}^{K} \mathsf{i}_k X_i,$$

and the corresponding projection functions,

$$\varpi_{\mathsf{i}_k} : \times i \in \cup_{k'=k}^{K} \mathsf{i}_k X_i \to \times i \in \cup_{k'=k+1}^{K} \mathsf{i}_k X_i.$$

Disintegrating a joint probability distribution with respect to this sequence of projection functions results in a sequence of conditional probability kernels

$$\pi(\mathsf{x}_{\mathsf{i}_k} \mid x_{\cup_{k'=k+1}^{K} \mathsf{i}_k})$$

and one tailing marginal probability distribution,

$$\pi(\mathsf{x}_{\mathsf{i}_K}).$$

I will refer to this sequence of probabilistic objects, or the partition of component indices that implicitly defines it, as a **conditional decomposition**.

Every ordered partition of the component spaces defines a unique conditional decomposition. For example, when working with a three-component product space

$$X^{1:3} = X_1 \times X_2 \times X_3,$$

the index partitions

$$\begin{aligned}
&\{1,2,3\}; \quad \{1,2\},\{3\}; \quad \{3\},\{1,2\}; \\
&\{1\},\{2,3\}; \quad \{2,3\},\{1\}; \quad \{1,3\},\{2\}; \\
&\{2\},\{1,3\}; \quad \{1\},\{2\},\{3\}; \quad \{1\},\{3\},\{2\}; \\
&\{2\},\{1\},\{3\}; \quad \{2\},\{3\},\{1\}; \quad \{3\},\{1\},\{2\}; \\
&\{3\},\{2\},\{1\}
\end{aligned}$$

all define a unique sequence of conditional probability kernels and tailing marginal distribution.

Each of these conditional decompositions can be more or less useful in different applications. For instance, some might be more interpretable in the context of a given applications, while others might facilitate the evaluation of certain probabilistic operations. We will return to the problem of identifying the useful decompositions in probabilistic modeling applications in future chapters.

Unfortunately, the number of possible conditional decompositions can be overwhelming. If we have I component spaces, then we can construct $H_d(I)$ distinct conditional decompositions, where $H_d(n)$ is the nth ordered Bell number (Knopfmacher and Mayes 2005). The ordered Bell numbers grow wildly fast as n increases (Figure 1).

That said, most practical applications consider only **atomic decompositions** that decompose joint probability distributions as much as possible. An atomic decomposition is defined by an atomic partition, where each cell consists of only a single component index,

$$\mathsf{i}_k = \{i_k\}.$$

This results in the sequence of conditional probability kernels

$$\pi(\mathsf{x}_{i_k} \mid x_{i_{k+1}}, \ldots, x_{i_I})$$

and the marginal distribution

$$\pi(\mathsf{x}_{i_I}).$$

Here the conditional decomposition "peels" off one component space at a time in the prescribed order.

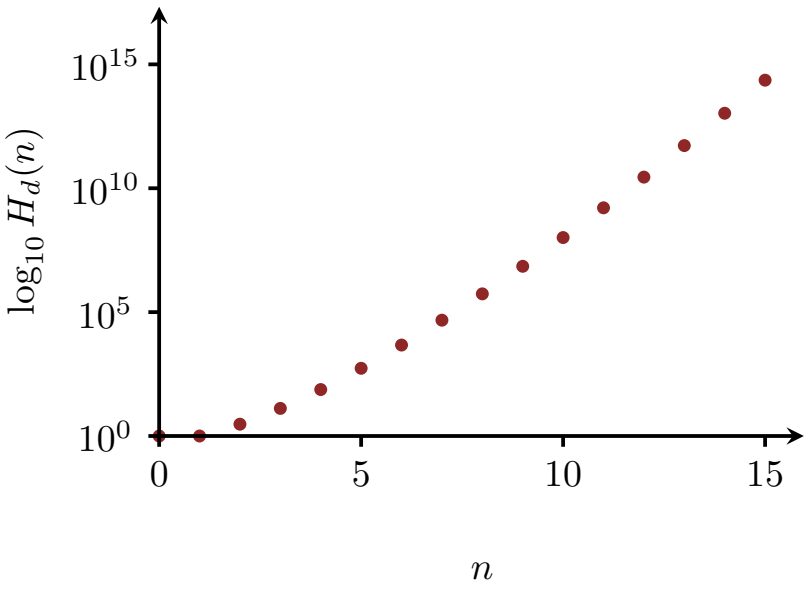


Figure 1: The nth ordered Bell number counts the number of ordered partitions of a finite set with n elements. As the size of a set increases, the number of ordered partitions increases extraordinarily quickly; even on a log scale the growth is exponential.

Given I component spaces, there are $I!$ of these atomic decompositions, each corresponding to a particular ordering of the components. For instance, when working with the product space

$$X^{1:3} = X_1 \times X_2 \times X_3,$$

we have only six unique atomic decompositions corresponding to the partitions

$$\begin{aligned} &1,2,3; \quad 1,3,2; \quad 2,1,3; \\ &2,3,1; \quad 3,1,2; \quad 3,2,1. \end{aligned}$$

2.3 Sequential Construction of Joint Probability Distributions

Often conditional decompositions are much more useful for building up joint distributions rather than tearing them down.

Given a partition of the component indices, we can always construct a joint distribution by working through the index subsets in reverse. We begin by engineering an appropriate marginal distribution

$$\pi(\mathsf{x}_{\mathsf{i}_K}).$$

over the space X^{i_K}. Then we engineer a conditional probability kernel,

$$\pi(\mathsf{x}_{\mathsf{i}_{K-1}} \mid x_{\mathsf{i}_k})$$

over the cross sections

$$X^{\mathsf{i}_{K-1}} \times \{x_{\mathsf{i}_K}\}$$

before iterating and engineering a conditional probability kernel

$$\pi(\mathsf{x}_{\mathsf{i}_k} \mid x_{\cup_{k'=k+1}^{K} \mathsf{i}_k})$$

for each subsequent cross section,

$$X^{\mathsf{i}_k} \times \{x_{\cup_{k'=k+1}^{K} \mathsf{i}_k}\}.$$

This sequence of conditional probability kernels "lifts" the initial marginal distribution into a joint distribution over the full product space.

When the index subsets are small, the intermediate cross sections will be low-dimensional spaces relative to the full product space. Because engineering probabilistic objects over low-dimensional spaces is typically much more manageable then trying to work with the entire product space at once, this piece-wise construction is often much more productive in practical applications.

When building off of an atomic decomposition, we will only ever have to work with a single active component space at a time, starting with

$$\pi(\mathsf{x}_{i_I}),$$

and then proceeding to each

$$\pi(\mathsf{x}_{i_k} \mid x_{i_{k+1}}, \ldots, x_{i_I}.)$$

2.4 Conditional Dependencies and Independencies

The behavior of the conditional probability distributions at any step of a conditional decomposition,

$$\pi(\mathsf{x}_{\mathsf{i}_k} \mid \tilde{x}_{\cup_{k'=k+1}^{K} \mathsf{i}_k})$$

can, in general, depend on the behavior in each of the remaining component spaces, X_i for

$$i \in \cup_{k'=k+1}^{K} \mathsf{i}_k.$$

If the conditional behavior in X^{i_k} varies with the behavior in the component space X_i, then we say that X^{i_k} is **conditionally dependent** on X_i.

Note that this is a *direct* dependency. The conditional behavior in X^{i_k} may not directly depend on the behavior in a component space X_i. If, however, it depends on the behavior in another component space $X_{i'}$, and the conditional behavior in that space depends on X_i, then the behavior in X_i can *indirectly* moderate the behavior in X^{i_k}.

These indirect dependencies are characterized by the **conditional independencies** of a conditional decomposition. For much more on conditional independencies in atomic conditional decompostions, see (Bishop 2006).

The conditional dependencies of a joint probability distribution are directly communicated by the structure of each conditional probability kernel in a conditional decomposition. For example, if

$$\pi(\mathsf{x}_{\mathsf{i}_k} \mid \tilde{x}_{\cup_{k'=k+1}^{K} \mathsf{i}_k}) = \pi(\mathsf{x}_{\mathsf{i}_k} \mid \tilde{x}_{i'})$$

then the behavior in X^{i_k} conditionally depends on the behavior in only the component space $X_{i'}$.

Alternatively, we can also communicate these conditional dependencies with an **dependency subset** d_k that includes all of the relevant component indices. In general,

$$\mathsf{d}_K = \emptyset$$

and

$$\mathsf{d}_k \subseteq \cup_{k'=k+1}^{K} \mathsf{i}_k.$$

Dependency subsets, however, are most informative when they include only fewer component indices. For instance, in the above example we would have $\mathsf{d}_k = \{i'\}$.

Conveniently, dependency subsets can be neatly organized into the component index partition that defines a conditional decomposition in the first place. In particular, the full structure of a conditional decomposition can be characterized by ordered pairs of index subsets,

$$((\mathsf{i}_1, \mathsf{d}_1), \ldots, (\mathsf{i}_K, \mathsf{d}_K),$$

where i_k communicates the active components at each step while d_k communicates the direct dependencies.

One nice feature of this more abstract characterization of conditional dependencies is it does not depend on the precise form of the conditional probability kernels. This allows us to, for instance, refer to any joint model that exhibits the same conditional dependencies. Even better, it allows us to separate the construction of a joint model into more steps: first specifying conditional dependencies, and only afterwards worrying about engineering the precise form of the conditional probability kernels.

To demonstrate the use of these more abstract conditional dependencies, let's consider the simplest possible behavior. If

$$\mathsf{d}_k = \emptyset$$

for all k, then the behavior of every conditional probability kernel will be completely independent of the behavior in every subsequent component space,

$$\pi(\mathsf{x}_{\mathsf{i}_k} \mid \tilde{x}_{\cup_{k'=k+1}^{K} \mathsf{i}_k}) = \pi(\mathsf{x}_{\mathsf{i}_k} \mid \emptyset) = \pi(\mathsf{x}_{\mathsf{i}_k}).$$

Any joint distribution that exhibits these conditional dependencies, however, is an independent product of component probability distributions in each X^{i_k}! When this is true for an atomic conditional decomposition, then these joint distributions will be independent product distributions,

$$\pi(\mathrm{d}x_1, \ldots, \mathrm{d}x_I) = \pi(\mathrm{d}x_1) \ \ldots \ \pi(\mathrm{d}x_i) \ \ldots \ \pi(\mathrm{d}x_I).$$

In other words, conditional dependencies give us yet another way to characterize independent product distributions.

All of this said, these abstract conditional dependencies can still be a bit ungainly to work with in practice. Fortunately, they admit a powerful *graphical* representation that we will discuss in **Section Blah**.

3 Product Probability Density Functions

In order to anchor all of this probability theory into a concrete foundation, we need to be able to actually implement probabilistic calculations. To do that, we'll need to consider probability density functions. Fortunately, product structure provides all of the ingredients we need to construct useful joint and conditional probability density functions.

3.1 Product Reference Measures

In theory, we can always try to define a reference measure directly over a given product space. As with most product structures, however, it will be almost always be more useful to build up a product reference measure from component pieces. If each component space X_i is equipped with a reference measure ν_i that admits practical integration, then their independent product $\nu^{1:I}$ defines a natural reference measure over the corresponding product space.

Provided that each of the component reference measures are σ-finite, the Fubini-Tonelli theorem then allows us to evaluate integrals with respect to $\nu^{1:I}$ by iterating the integration operations defined by each ν_i. As we discussed in **Section Blah**, for instance, integrals with respect to a product counting measure can be evaluated by nesting discrete sums, while integrals with respect to a product of Lebesgue measures can often be evaluated by nesting one-dimensional Riemann integrals.

Another powerful feature of independent product reference measures is their disintegrations with respect to any projection function are as well-behaved as we could hope.

Consider, for example, a subset of components,

$$\mathsf{i} \subset 1 : I,$$

the corresponding projection function,

$$\varpi_{\mathsf{i}} : X^{1:I} \to X^{\mathsf{i}},$$

and a natural reference measure over the output space, ν^{i}. Together, these objects disintegrate $\nu^{1:I}$ into a collection of conditional measures

$$\nu^{\varpi_{\mathsf{i}}}(\mathsf{x}_{\mathsf{i}^c} \mid x_{\mathsf{i}})$$

defined over each of the cross sections

$$\varpi_{\mathsf{i}}^{-1}(x_{\mathsf{i}}) = X^{\mathsf{i}^c} \times \{x_{\mathsf{i}}\}.$$

Moreover, these conditional measures are equivalent to independent product measures over the complementary components,

$$\nu^{\varpi_{\mathsf{i}}}(\mathsf{x}_{\mathsf{i}^c} \mid \tilde{x}_{\mathsf{i}}) \cong \nu^{\mathsf{i}^c}(\mathsf{x}_{\mathsf{i}^c})$$

for all

$$\tilde{x}_{\mathsf{i}} \in X^{\mathsf{i}}.$$

This means that we can also use the Fubini-Tonelli theorem to evaluate *conditional* integrals as nested component integrals.

If we know how to integrate on each component space, then independent product reference measures allow us to integrate on all of the relevant product spaces and cross sections.

3.2 Joint and Conditional Probability Density Functions

Once we have an independent product reference measure, the construction of probability density functions, and consequently the evaluation of expectation values, is immediate.

For any probability distribution π defined over the full product space, we can construct the **joint probability density function**, or simply **joint density function** using the independent product reference measure,

$$p(x_1, \ldots, x_I) = \frac{\mathrm{d}\pi}{\mathrm{d}\nu^{1:I}}(x_1, \ldots, x_I).$$

Given this joint density function, we can then compute joint expectation values by nesting component integrals, for example

$$\begin{aligned}
\mathbb{E}_\pi[g] = \int \nu_1(\mathrm{d}x_1) \Bigg[& \ldots \\
& \int \nu_i(\mathrm{d}x_i) \Bigg[\ldots \\
& \qquad \int \nu_I(\mathrm{d}x_I)\, p(x_1, \ldots, x_I)\, g(x_1, \ldots, x_I) \\
& \qquad \ldots \Bigg] \\
& \ldots \Bigg].
\end{aligned}$$

Provided that the joint expectation value is well-defined, any ordering of the component integrals will yield the same answer.

Similarly, for any $\mathsf{i} \subset 1:I$ we can construct a **marginal probability density function**, or simply **marginal density function**,

$$p(x_\mathsf{i}) = \frac{\mathrm{d}(\varpi_\mathsf{i})_* \pi}{\mathrm{d}\nu^\mathsf{i}}(x_\mathsf{i}),$$

On the cross sections of any i we can define **conditional probability density functions**, or simply **conditional density functions**,

$$p(x_{\mathsf{i}^c} \mid x_\mathsf{i}) = \frac{\mathrm{d}\pi^{\varpi_\mathsf{i}}}{\mathrm{d}\nu^{\varpi_\mathsf{i}}}(x_{\mathsf{i}^c} \mid x_\mathsf{i}).$$

The computation of marginal and conditional expectation values once again follows from the Fubini-Tonelli theorem and the component integration operations.

This is all a bit easier to follow when we allow for explicit components. To that end, consider $I = 4$ rigid real lines that form the product space

$$X^{1:I} = \mathbb{R}^4.$$

Because each component space is equipped with a specific metric, we can construct a unique Lebesgue reference measure λ_i on each of them. The independent product of these component reference measures defines the joint reference measure $\lambda^{1:4}$.

This allows us to construct joint density functions,

$$p(x_1, x_2, x_3, x_4) = \frac{\mathrm{d}\pi}{\mathrm{d}\nu^{1:I}}(x_1, x_2, x_3, x_4),$$

and evaluate joint expectation values with nested Riemann integrals, for instance

$$\mathbb{E}_{\pi}[g] = \int \mathrm{d}x_1 \int \mathrm{d}x_2 \int \mathrm{d}x_3 \int \mathrm{d}x_4 \, p(x_1, x_2, x_3, x_4) \, g(x_1, x_2, x_3, x_4).$$

Given the component subset

$$\mathsf{i} = \{2, 4\},$$

with

$$\mathsf{i}^c = \{1, 3\},$$

we can disintegrate this joint density function into a marginal density function,

$$p(x_2, x_4),$$

and conditional density functions,

$$p(x_1, x_3 \mid x_2, x_4).$$

Marginal and conditional integrals are, once again, given by nesting the appropriate component integrals.

3.3 Decomposition and Composition

In practice, we rarely if ever construct joint, marginal, and conditional density functions separately. Instead, it's almost always more convenient *derive* them from each other.

As we saw in Chapter 9, for example, a marginal density function can be derived from the a joint density function by integrating over the cross sections,

$$p(x_{\mathsf{i}}) \overset{\nu^{\mathsf{i}}}{=} \int \nu^{\varpi_{\mathsf{i}}}(x_{\mathsf{i}^c}) p(x_{\mathsf{i}^c}, x_{\mathsf{i}}).$$

This is no longer an abstract result, however, as the conditional integrals can now be evaluated with the Fubini-Tonelli theorem.

At the same time, the product rule relates the joint, marginal, and conditional probability density functions together,

$$p(x_{\mathsf{i}^c}, x_{\mathsf{i}}) \overset{\nu^{1:I}}{=} p(x_{\mathsf{i}^c} \mid x_{\mathsf{i}}) \, p(x_{\mathsf{i}}).$$

This allows us to derive any one of these objects from the other two.

Given a joint density function we can derive a marginal density function by integrating over the cross sections, and then construct conditional density functions from point-wise division,

$$p(x_{\mathrm{i}^c} \mid x_{\mathrm{i}}) \overset{\nu^{1:I}}{=} \frac{p(x_{\mathrm{i}^c}, x_{\mathrm{i}})}{p(x_{\mathrm{i}}).}$$

To demonstrate, let's consider the product space $X^{1:2} = \mathbb{R}^2$ equipped with a multivariate Lebesgue reference measure. This allows us to define a joint distribution through the joint density function (Figure 2a)

$$p(x_1, x_2) = \frac{1}{2\,\pi\,\sqrt{\sigma_1^2\,\sigma_2^2\,(1 - \rho^2)}} \exp\left[-\frac{1}{2} Q(x_1, x_2)\right]$$

where

$$Q(x_1, x_2) = \frac{\sigma_2^2\,x_1^2 - 2\,\rho\,\sigma_1\,\sigma_2\,x_1\,x_2 + \sigma_1^2\,x_2^2}{\sigma_1^2\,\sigma_2^2\,(1 - \rho^2).}$$

The marginal density function over X_1 is given by integrating the joint density function over the level sets of ϖ_1, which are just lines where x_1 is fixed (Figure 2b),

$$p(x_1) = \int \mathrm{d}x_2\, p(x_1, x_2).$$

After some calculus, which I've sequestered to Appendix A.1, becomes a normal density function (Figure 2c)

$$p(x_1) = \mathrm{normal}(x_1 \mid 0, \sigma_1).$$

Similarly, the conditional density functions become normal density functions of their own (Figure 2d).

$$p(x_2 \mid x_1) = \frac{p(x_1, x_2)}{p(x_1)}$$
$$\mathrm{normal}\left(x_2 \,\middle|\, \rho\frac{\sigma_2}{\sigma_1} x_1, \sigma_2\sqrt{1 - \rho^2}\right).$$

Because of the symmetry of the joint density function, the same calculations apply when deriving the marginal density over X_2 and the corresponding conditional density functions. The result there just swaps all of the component indices (Figure 3)

$$p(x_2) = \mathrm{normal}(x_2 \mid 0, \sigma_2)$$
$$p(x_1 \mid x_2) = \mathrm{normal}\left(x_1 \,\middle|\, \rho\frac{\sigma_1}{\sigma_2} x_2, \sigma_1\sqrt{1 - \rho^2}\right).$$

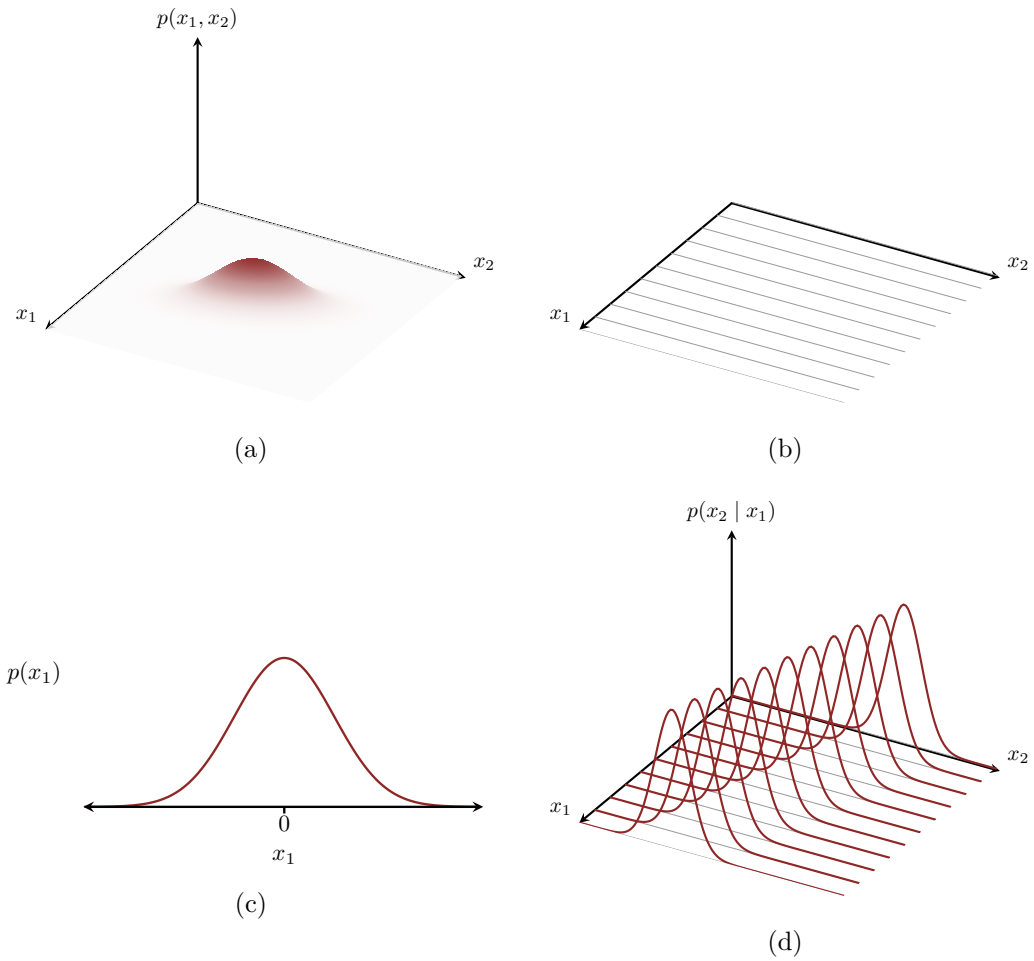


Figure 2: The two-dimensional real space $\mathbb{R}^2$ is a product space with rich product structure. (a) A natural two-dimensional Lebesgue reference measure allows us to define joint distributions by joint density functions. (b) Distinguishing the first component x_1 defines a projection function ϖ_1 whose level sets are cross sections where x_1 is fixed. (c) The marginal density function with respect to this projection function is given by integrating the joint density function over the level sets of ϖ_1. (d) The remaining information about the joint density function is encoded in conditional density functions defined over each level set.

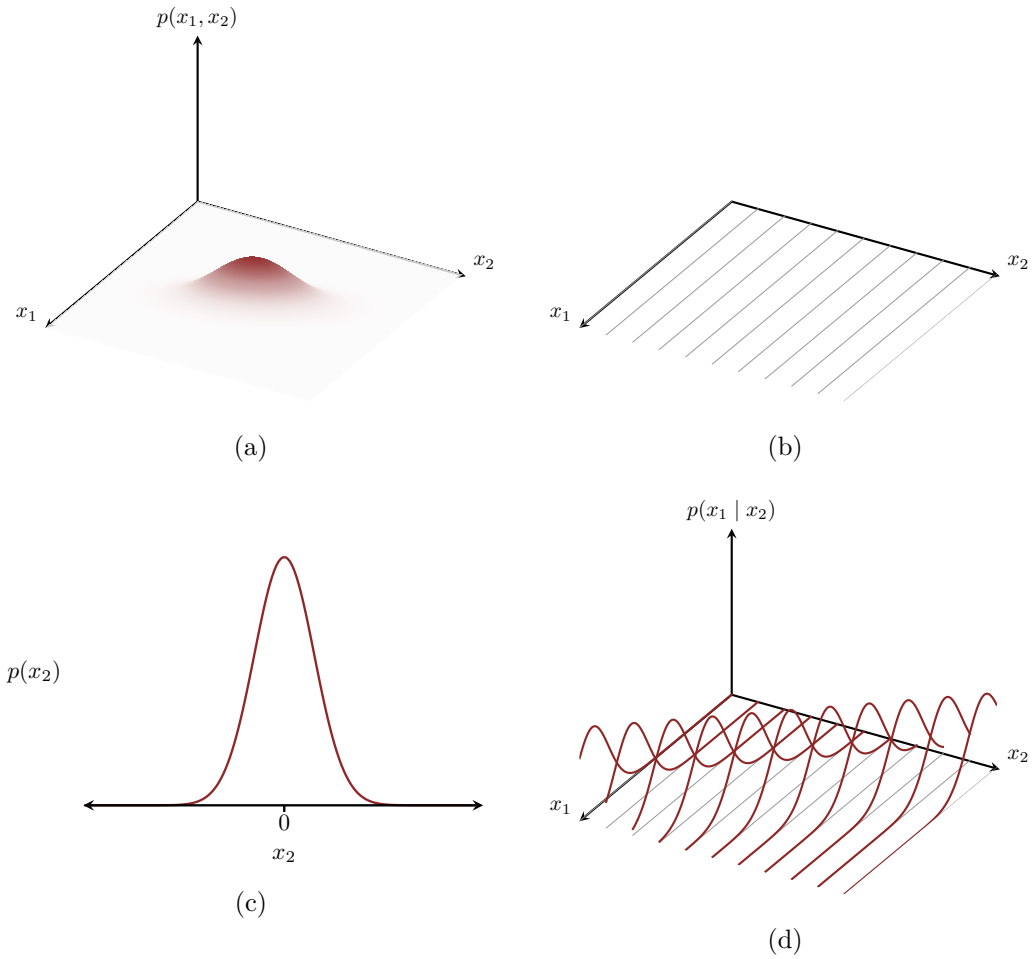


Figure 3: We can also disintegrate the joint density function in Figure 3 with respect to ϖ_2.

Alternatively, a marginal density function and collection of conditional density functions together define joint density function by point-wise multiplication,

$$p(x_{\mathsf{i}^c}, x_{\mathsf{i}}) \overset{\nu^{1:I}}{=} p(x_{\mathsf{i}^c} \mid x_{\mathsf{i}})\, p(x_{\mathsf{i}}).$$

Moreover, given a conditional decomposition with the active component subsets i_k and a sequence of conditional probability density functions, we can apply the product rule iteratively to define the joint probability density function

$$p(x_{1:I}) \overset{\nu^{1:I}}{=} \prod_{k=1}^{K-1} \left[p(x_{\mathsf{i}_k} \mid x_{\cup_{k'=k+1}^{K} \mathsf{i}_k}) \right] p(\mathsf{i}_K).$$

For example, given the product space

$$X^{1:I} = \mathbb{R}^4,$$

the conditional decomposition

$$((\{4\}, \{1,2,3\}), (\{1,3\}, \{2\}), (\{2\}, \{\emptyset\})$$

would yield the joint density function

$$p(x_1, x_2, x_3, x_4) = p(x_4 \mid x_1, x_2, x_3)\, p(x_1, x_3 \mid x_2)\, p(x_2).$$

Even more explicitly, consider the mixed product space

$$X^{1:2} = \mathbb{R} \times \mathbb{Z}$$

and the conditional decomposition

$$((\{2\}, \{1\}), (\{1\}, \{\emptyset\}).$$

Taking

$$p(x_1) \propto x_1^2\, (1 - x_1)^2$$

and

$$p(x_2 \mid x_1) \propto x_1^{x_2}\, (1 - x_1)^{10 - x_2}$$

gives the joint density function (Figure 17)

$$\begin{aligned} p(x_1, x_2) &= p(x_2 \mid x_1)\, p(x_1) \\ &\propto x_1^{2 + x_2}\, (1 - x_1)^{12 - x_2}. \end{aligned}$$

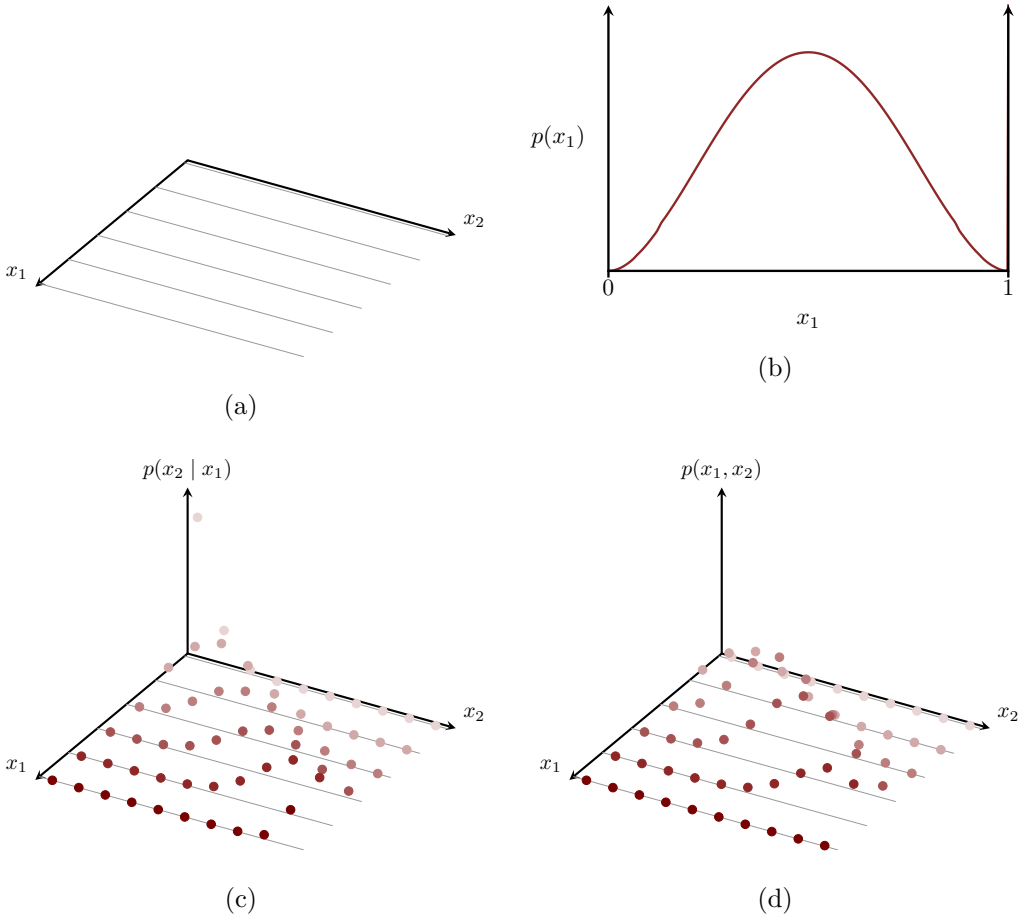


Figure 4: Conditional decompositions provide a foundation on which we can build up joint density functions, and hence joint distributions. (a) A choice of conditional decomposition defines a marginal space, here X_1, and sequence of cross sections, here just $X_2 \times \{x_2\}$. (b) Any choice of marginal density function and (c) conditional density functions defines (d) a joint density function over the full product space.

3.4 The Robustness of Logarithmic Density Functions

All of this said, evaluating a joint probability density function defined by a product of conditional density functions on a computer can be surprisingly awkward. In particular, the multiplication of many sequential conditional density function evaluations, each of which can result in a number close to zero, is prone to numerical underflow.

One convenient way to avoid these numerical issues, is to not work with probability density functions but rather work with their composition with the natural logarithm function,

$$p(x_1, \ldots, x_I) \mapsto \log \circ p(x_1, \ldots, x_I).$$

The logarithm of the joint density function is then given by *adding* together the logarithm of the conditional density functions,

$$\log \circ p(x_{1:I}) \overset{\nu^{1:I}}{=} \sum_{k=1}^{K-1} \left[\log \circ p(x_{\mathsf{i}_k} \mid x_{\cup_{k'=k+1}^{K} \mathsf{i}_k}) \right] + \log \circ p(\mathsf{i}_K).$$

Because addition is a much more numerically stable operation than multiplication, $\log \circ p(x_{1:I})$ is much easier to evaluate accurately on a computer.

Once we've computed this logarithmic density function, we can always recover the joint density function whenever needed by applying the exponential function,

$$p(x_1, \ldots, x_I) = \exp\left(\log \circ p(x_1, \ldots, x_I)\right).$$

3.5 Conditioning and Partial Evaluation

The product rule allows us to immediately condition a joint probability density function on the cross sections defined by any subset of components i,

$$p(x_{\mathsf{i}^c} \mid x_{\mathsf{i}}) \overset{\nu^{1:I}}{=} \frac{p(x_{\mathsf{i}^c}, x_{\mathsf{i}})}{p(x_{\mathsf{i}}).}$$

In practice, however, the application of this equation is often limited by our ability to calculate the denominator,

$$p(x_{\mathsf{i}}) \overset{\nu^{\mathsf{i}}}{=} \int \nu^{\varpi_{\mathsf{i}}}(x_{\mathsf{i}^c}) p(x_{\mathsf{i}^c}, x_{\mathsf{i}}).$$

for each $x_{\mathsf{i}} \in X^{\mathsf{i}}$.

That said, when we bind x_{i} to a particular value $\tilde{x}_{\mathsf{i}}$, the denominator reduces to just some number

$$p(\tilde{x}_{\mathsf{i}}) \in \mathbb{R}^+.$$

In this case, the *particular* conditional density function is, up to proportionality, equal to the partial evaluation of the joint density function

$$p(x_{\mathsf{i}^c} \mid \tilde{x}_{\mathsf{i}}) \overset{\nu^{1:I}}{=} \frac{p(x_{\mathsf{i}^c}, \tilde{x}_{\mathsf{i}})}{p(\tilde{x}_{\mathsf{i}}).} \propto p(x_{\mathsf{i}^c}, \tilde{x}_{\mathsf{i}}).$$

Consequently, we don't need to be able to evaluate the marginal density function in order to derive *unnormalized* conditional density functions. This ends up being a surprisingly powerful shortcut in many practical applications.

Consider, for instance, a joint density function over the binary product space $X_1 \times X_2$ defined by

$$p(x_1, x_2) \overset{\nu^{1:2}}{=} p(x_2 \mid x_1)\, p(x_1).$$

Because we specify $p(x_2 \mid x_1)$ directly, any application that needs a conditional density function $p(x_2 \mid \tilde{x}_1)$ is immediately satisfied.

The same, however, cannot be said for any application that needs an opposing conditional density function, $p(x_1 \mid \tilde{x}_2)$. This requires being able to calculate

$$\begin{aligned} p(x_1 \mid \tilde{x}_2) &= \frac{p(x_1, \tilde{x}_2)}{p(\tilde{x}_2)} \\ &= \frac{p(x_1, \tilde{x}_2)}{\int \nu_1(\mathrm{d}x_1) p(x_1, \tilde{x}_2).} \end{aligned}$$

If we cannot evaluate the normalizing integral, then we cannot construct $p(x_1 \mid \tilde{x}_2)$. That said, if the normalization is irrelevant, then we can skip the normalizing integral entirely,

$$p(x_1 \mid \tilde{x}_2) \overset{\nu^{1:I}}{\propto} p(x_1, \tilde{x}_2).$$

One circumstance where the marginal density function can't be neglected is when we want to compare two conditional distributions to each other. The Radon-Nikodym derivative between two conditional distributions is given by the ratio of their conditional density functions. Using the product rule, we can write this ratio as

$$\begin{aligned} \frac{p(x_1 \mid \tilde{x}_2)}{p(x_1 \mid \tilde{x}'_2)} &\overset{\nu^{1:2}}{=} \frac{p(x_1, \tilde{x}_2)}{p(\tilde{x}_2)} \frac{p(\tilde{x}'_2)}{p(x_1, \tilde{x}'_2)} \\ &\overset{\nu^{1:2}}{=} \frac{p(x_1, \tilde{x}_2)}{p(x_1, \tilde{x}'_2)} \frac{p(\tilde{x}'_2)}{p(\tilde{x}_2).} \end{aligned}$$

In this case, the ratio of the marginal density function evaluations is critical to an accurate comparison.

4 Directed Graphical Models

So far we have seen conditional decompositions defined explicitly, as a collection of active and dependency subsets, and implicitly, by writing out a sequence of conditional density functions and tailing marginal density function. In this section we will be introduced to a *visual* representation of conditional decompositions that neatly complements these methods.

4.1 Directed Acyclic Graphs

A **graph** is a collection of **nodes** with **edges** connected certain pairs of nodes. Typically nodes represent mathematical objects and edges the relationships, or lack thereof, between those objects.

Directed graphs (Bishop 2006) distinguish between edges that originate from one node and terminate in another, and vice versa. The structure of a directed graph is particularly straightforward to visualize, with two-dimensional shapes representing each node and one-dimensional arrows the directed edges between them (Figure 5).

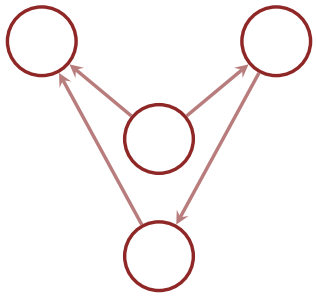


Figure 5: Directed graphs are typically visualized using shapes to represent each node and arrows to represent the edges between them. When the nodes of a directed graph are labeled, we can also label the shapes accordingly.

The nodes at which a directed edge originates is referred to as the **parent** of the node at which the edge terminates. Similarly, the terminating node is referred to as a **child** of originating node (Figure 6). This familial language motivates a variety of related vocabulary, such as referring to two nodes that share a common parent as siblings.

A sequence of directed edges connecting parent nodes to child nodes defines a **path** through the graph. In general, these paths can circle back on themselves, forming **cycles** (Figure 7a). Directed graphs that do not exhibit any cycles (Figure 7b) are known as **directed acyclic graphs**, or often just **DAGs** for short.

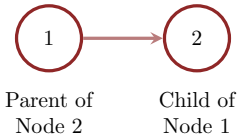
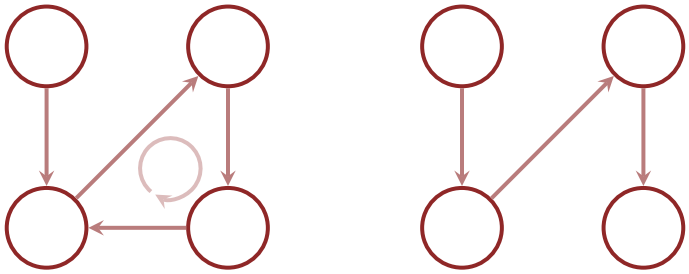


Figure 6: Two nodes connected by a directed edge are referred to as the parent and child of each other. This terminology allows us to compact describe relationships between nodes in a directed graph.



(a) (b)

Figure 7: (a) Cycles in a directed graph are formed by paths of edges that connect back on themselves. (b) In a directed acyclic graph, each path originates and terminates at a distinct node.

Because of the lack of cycles, directed acyclic graphs exhibit a well-defined notion of directionality (Figure 8). On one side of the graph we have **root nodes** with no parents, and on the other we have **leaf nodes** with no children.

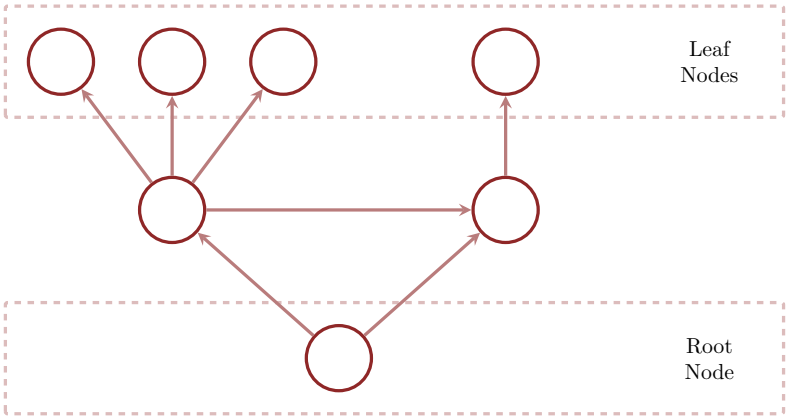


Figure 8: Directed acyclic graph can be arranged in a natural direction, starting with root nodes who have no parents and leaf nodes who have no children.

Although visualizations of directed graphs don't technically have to respect this directionality, they tend to be more effective when they do. That said, there is no universal convention for whether root notes should be on the top, bottom, left, or right of a visualization. We are free to pick whichever convention is most convenient, provided that we maintain that convention consistently.

4.2 Graphical Models of Atomic Conditional Decompositions

Conveniently, every atomic conditional decomposition defines a unique directed acyclic graph, which can then be used to visualize the conditional structure. A conditional decomposition represented by a directed acyclic graph is known as a **directed graphical model**, or sometimes just **DGM** for short.

Consider, for example, the atomic conditional decomposition

$$((\{i_1\}, \mathsf{d}_1), \quad \ldots, \quad (\{i_I\}, \mathsf{d}_I)).$$

Each subset of active components,

$$\mathsf{i}_k = \{i_k\}$$

can be represented by a node labeled with the corresponding component variable. At the same time, each pair $((\{i_k\}, \mathsf{d}_k)$ defines a collection of directed edges, each of which originates from

one of the nodes representing the elements of d_k and terminating at the node representing i_k.

To demonstrate, let's build a graphical model for a joint distribution defined over the product space

$$X = X_1 \times X_2 \times X_3 \times X_4$$

with the conditional decomposition

$$((\{4\}, \{1, 3\}), \quad (\{3\}, \{1, 2\}), \quad (\{1\}, \{2\}), \quad (\{2\}, \{\emptyset\})).$$

Equivalently, we can interpret this as building a graphical model for the density function decomposition

$$\begin{aligned} p(x_1, x_2, x_3, x_4) = \; & p(x_4 \mid x_1, x_3) \\ & \cdot p(x_3 \mid x_1, x_2) \\ & \cdot p(x_1 \mid x_2) \\ & \cdot p(x_2). \end{aligned}$$

Each of the four component spaces defines a node in the corresponding graphical model (Figure 9a). Because the behavior in X_4 depends on the behavior in X_1 and X_3, we then place two directed edges from the nodes representing X_1 and X_3 to the node representing X_4 (Figure 9b). Next, we place two directed edges that terminate at the node representing X_3, and one that terminates at the node representing X_1 (Figure 9c). Finally, we place one last directed edge from the node representing X_2 to the node representing X_1 (Figure 9d). The behavior in X_2 is independent of the behavior in any of the other component spaces; consequently, there are no more directed edges to place.

For this conditionald decomposition, the node representing X_4 is the lone leaf node. At the same time, the node representing X_2 is the only root node.

The more coupled the behavior in the component spaces are to each other, the more directed edges the corresponding graphical model will feature. Often the most informative aspect of a graphical model is the *absence* of directed edges.

4.3 Graphical Models of More General Conditional Decompositions

Unfortunately, once we move away from atomic conditional decompositions, graphical modeling becomes a little bit more complicated. The problem is that not every general conditional decompositions defines a distinct directed acyclic graph. In other words, some graphical models will be ambiguous.

To see what can go wrong, let's go back to the product space

$$X = X_1 \times X_2 \times X_3 \times X_4,$$

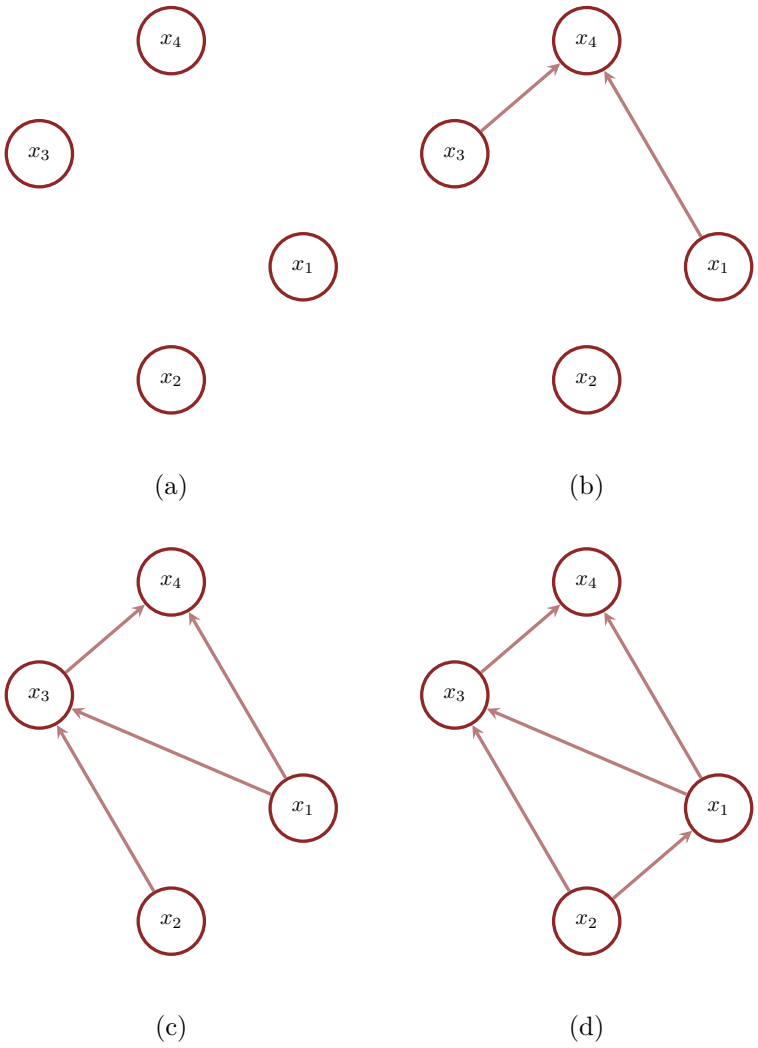


Figure 9: Graphical models are straightforward to build for atomic conditional decompositions. (a) After placing nodes for each component space, (b-d) we work down the conditional decomposition, placing directed arrows between every component node and its dependencies.

but now consider the conditional decomposition

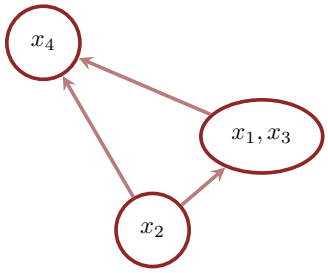
$$((\{4\}, \{1, 2\}), \quad (\{1, 3\}, \{2\}), \quad (\{2\}, \{\emptyset\})).$$

Again, we can equivalently interpret this as the density function decomposition

$$p(x_1, x_2, x_3, x_4) = p(x_4 \mid x_1, x_2)\, p(x_1, x_3 \mid x_2)\, p(x_2).$$

Here we'll need a directed edge to represent the conditional dependence of the behavior in X_4 on the behavior in X_1. There is no node, however, that represents just X_1. Instead we have a node that represents both X_1 and X_3.

One option is to add a directed edge that originates from the node representing $X_1 \times X_3$ and terminates at the node representing X_4 (Figure 10). This approach, however, would have us add the same exact edge if the behavior in X_4 conditionally depended on X_3 or $X_1 \times X_3$!



$$\begin{aligned} &p(x_1, x_2, x_3, x_4) \\ &\quad = p(x_4 \mid x_1, x_2)\, p(x_1, x_3 \mid x_2)\, p(x_2) \end{aligned}$$

$$\begin{aligned} &p(x_1, x_2, x_3, x_4) \\ &\quad = p(x_4 \mid x_3, x_2)\, p(x_1, x_3 \mid x_2)\, p(x_2) \end{aligned}$$

$$\begin{aligned} &p(x_1, x_2, x_3, x_4) \\ &\quad = p(x_4 \mid x_1, x_2, x_3)\, p(x_1, x_3 \mid x_2)\, p(x_2) \end{aligned}$$

Figure 10: Not every non-atomic conditional decomposition translates to a unique graphical model. The graphical model here is consistent with three different conditional decompositions.

In other words, this graphical model represents three distinct conditional decompositions,

$$\begin{aligned} &((\{4\}, \{1, 2\}), \quad (\{1, 3\}, \{2\}), \quad (\{2\}, \{\emptyset\})) \\ &((\{4\}, \{1, 3\}), \quad (\{1, 3\}, \{2\}), \quad (\{2\}, \{\emptyset\})) \\ &((\{4\}, \{1, 2, 3\}), \quad (\{1, 3\}, \{2\}), \quad (\{2\}, \{\emptyset\})), \end{aligned}$$

at the same time. Without additional information, there would be no way to know which of these conditional decomposition was intended from the graphical model alone.

Fortunately, it's not *too* difficult to characterize a more general class of conditional decompositions that correspond to unique graphical models. The key is to consider only those conditional decompositions where each dependency subset d_k is the union of some number of active component subsets $\mathsf{i}_{k'}$,

$$\mathsf{d}_k = \bigcup_{k'} \mathsf{i}_{k'}.$$

In this case, the behavior in X^{i_k} will always depend on *all* of the component spaces in a node representing $X^{\mathsf{i}_{k'}}$ or *none* of them. Consequently, each directed edge will always have a unique origin and terminus.

For instance, of the three conditional decompositions that we considered above, only

$$((\{4\}, \{1, 2, 3\}), \quad (\{1, 3\}, \{2\}), \quad (\{2\}, \{\emptyset\}))$$

satisfies this disambiguating criteria. The other two conditional decompositions would have to be excluded to avoid any ambiguity.

A simpler way to interpret this disambiguating criteria is that is reinterprets the components of the product space to be not the X_i,

$$X = \times_{i=1}^{I} X_i,$$

but rather the X^{i_k},

$$X = \times_{k=1}^{K} X^{\mathsf{i}_k}.$$

Any conditional decomposition built around these composite components will necessarily treat each X^{i_k} uniformly.

Many references avoid this awkwardness entirely by using graphical models for only atomic conditional decompositions. That said, more general graphical modeling can be extremely useful in practice, even if used only informally. In this book, I will be relatively generous with my applications of graphical modeling, taking care to respect the disambiguating criteria.

4.4 Bells and Whistles

The nodes and edges in directed graphical models convey a wealth of information. We can, however, pack even more information into directed graphical models by augmenting this basic graph structure with some special features.

4.4.1 Auxiliary Nodes

Many applications consider joint distributions whose behavior depends on the behavior of auxiliary spaces that are not equipped with any probabilistic structure of their own. In other words, these joint distributions are *parameterized* by the configuration of the auxiliary spaces.

These non-probabilistic dependencies can be represented in graphical models by introducing multiple types of nodes, one to represent the probabilistic spaces and one to represent the auxiliary spaces. I like to use *rounded* shapes to represent probabilistic nodes and *rectangular* shapes to represent auxiliary nodes.

For instance, the conditional structure of

$$\begin{aligned} p(x_1, x_2, x_3, x_4; y) = \; & p(x_4 \mid x_1, x_3) \\ & \cdot p(x_1 \mid x_3; y) \\ & \cdot p(x_3 \mid x_2) \\ & \cdot p(x_2) \end{aligned}$$

can be represented by the graphical model in Figure 11.

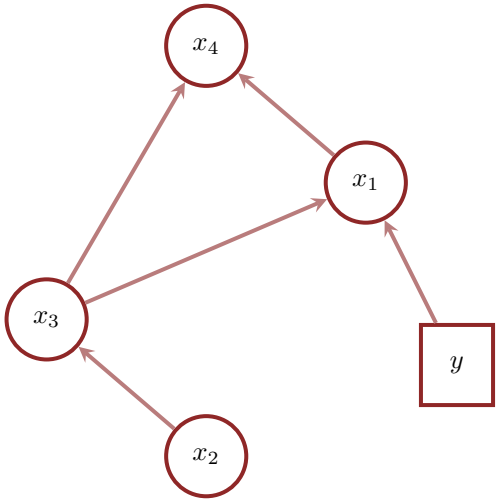


Figure 11: Rectangular nodes communicate non-probabilistic dependencies in joint probability distributions. Here the conditional density functions $p(x_1 \mid x_3; y)$ are parameterized by the configuration of a non-probabilistic space, $y \in Y$.

4.4.2 Pushforward Nodes

Sometimes the behavior of a component space depends on the behavior of other component spaces through only the output of some intermediate function. For instance, instead of

$$\begin{aligned} p(x_1, x_2, x_3, x_4; y) = \; & p(x_4 \mid x_1, x_3) \\ & \cdot p(x_1 \mid x_3; y) \\ & \cdot p(x_3 \mid x_2) \\ & \cdot p(x_2) \end{aligned}$$

we might have

$$\begin{aligned} p(x_1, x_2, x_3, x_4; y) = \; & p(x_4 \mid f(x_1, x_3)) \\ & \cdot p(x_1 \mid x_3; y) \\ & \cdot p(x_3 \mid x_2) \\ & \cdot p(x_2). \end{aligned}$$

In this case, the behavior in X_4 doesn't directly depend on the behavior in $X_1 \times X_3$, but rather the behavior in $X_1 \times X_3$ pushed forward through f.

One way to communicate these functional dependencies in a graphical model is to introduce an explicit variable for the function output,

$$\begin{aligned} p(x_1, x_2, x_3, x_4; y) = \; & p(x_4 \mid y = f(x_1, x_3)) \\ & \cdot p(x_1 \mid x_3; y) \\ & \cdot p(x_3 \mid x_2) \\ & \cdot p(x_2), \end{aligned}$$

and then represent that variable with its own node. To avoid any ambiguity, however, we'll need to decorate this node differently from the component nodes. I like to use dashed borders (Figure 12)

4.4.3 Conditioned Nodes

Some applications are less concerned with a given joint distribution than certain conditional distributions derived from that joint distribution when we bind particular component variables to particular values.

Once we have a graphical model for the joint distribution, this conditioning process is straightforward to visualize by adding a distinct decoration to the nodes corresponding to the bound variables. A common convention is to fully color these nodes.

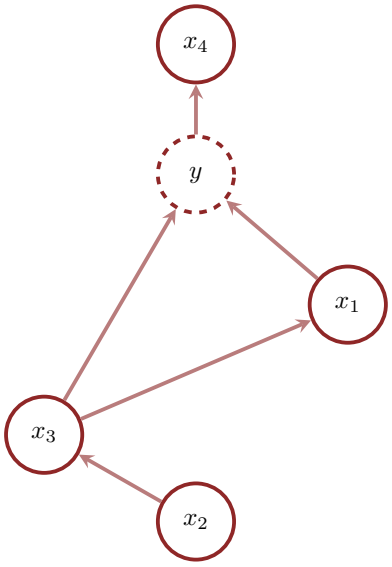
For example, let's say that we start with the joint distribution defined by the probability density functions (Figure 13a)

$$\begin{aligned} p(x_1, x_2, x_3, x_4) = \; & p(x_4 \mid x_1, x_3) \\ & \cdot p(x_1 \mid x_3) \\ & \cdot p(x_3 \mid x_2) \\ & \cdot p(x_2), \end{aligned}$$

but we are interested in the conditional distribution defined by

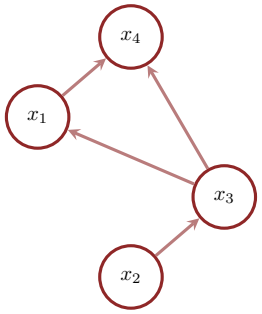
$$p(x_2, x_3 \mid \tilde{x}_1, \tilde{x}_4) = \frac{p(\tilde{x}_1, x_2, x_3, \tilde{x}_4)}{p(\tilde{x}_1, \tilde{x}_4).}$$

We can visualize this particular conditioning by coloring in the nodes representing X_1 and X_4 (Figure 13b).



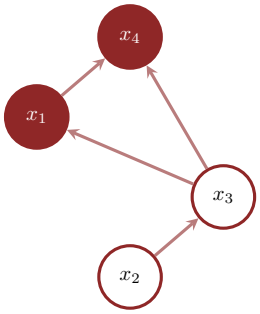
$$
\begin{aligned}
&p(x_1, x_2, x_3, x_4; y) \\
&\quad = \; p(x_4 \mid y = f(x_1, x_2)) \\
&\qquad \cdot p(x_1 \mid x_3) \\
&\qquad \cdot p(x_3 \mid x_2) \\
&\qquad \cdot p(x_2)
\end{aligned}
$$

Figure 12: Dashed, circular nodes represent the output of functions that depend on components variables. We can interpret these nodes as modeling the pushforward distribution along the intermediate function.



$$
\begin{aligned}
&p(x_1, x_2, x_3, x_4) \\
&\quad = \; p(x_4 \mid x_1, x_2) \\
&\qquad \cdot p(x_1 \mid x_3) \\
&\qquad \cdot p(x_3 \mid x_2) \\
&\qquad \cdot p(x_2)
\end{aligned}
$$

(a)



$$
\begin{aligned}
&p(x_2, x_3 \mid \tilde{x}_1, \tilde{x}_4) \\
&\quad \propto \; p(\tilde{x}_1, x_2, x_3, \tilde{x}_4) \\
&\quad \propto \; p(\tilde{x}_4 \mid \tilde{x}_1, x_2) \\
&\qquad \cdot p(\tilde{x}_1 \mid x_3) \\
&\qquad \cdot p(x_3 \mid x_2) \\
&\qquad \cdot p(x_2)
\end{aligned}
$$

(b)

Figure 13: The various conditional distributions that we can derive from a joint distribution can be represented by (a) taking the graphical model for the joint distribution and (b) coloring in the nodes representing the components being bound to particular values.

4.4.4 Plate Notation

Some conditional patterns are sufficiently common that it's useful to introduce graphical shorthands that allow for more compact visualizations.

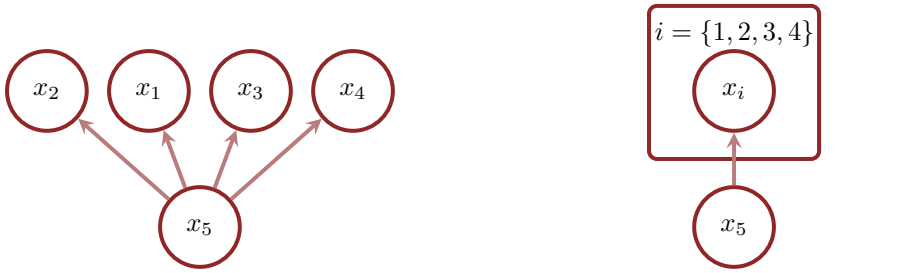
Consider, for example, the product space $X = X^{1:5}$, and the conditional decomposition

$$(\{\{1\},\{5\}),\quad (\{2\},\{5\}),\quad (\{3\},\{5\}),\quad (\{4\},\{5\}),\quad (\{5\},\{\emptyset\})),$$

or, equivalently, the sequence of probability density functions

$$\begin{aligned} p(x_1, x_2, x_3, x_4, x_5) = \; & p(x_1 \mid x_5) \\ & \cdot p(x_2 \mid x_5) \\ & \cdot p(x_3 \mid x_5) \\ & \cdot p(x_4 \mid x_5) \\ & \cdot p(x_5). \end{aligned}$$

This particular conditional structure results in a fan-like graphical model (Figure 14a), which can take up a lot of visual space. We can make the graphical model a bit more compact, however, by packing the first four nodes together into a single **plate** that represents their common dependence on the fifth node.



$p(x_1, x_2, x_3, x_4, x_5)$
$= \left[\prod_{i=1}^{4} p(x_i \mid x_5)\right] p(x_5)$

(a)

$p(x_1, x_2, x_3, x_4, x_5)$
$= \left[\prod_{i=1}^{4} p(x_i \mid x_5)\right] p(x_5)$

(b)

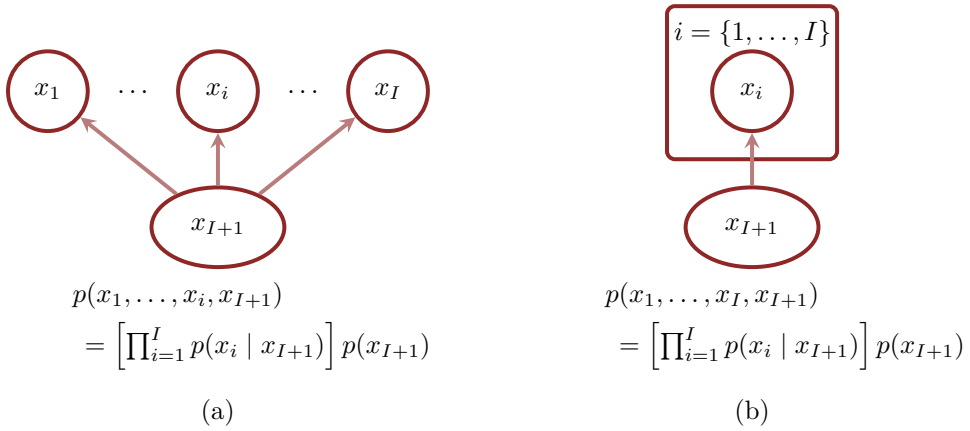
Figure 14: (a) Conditional dependencies where many subsets of active components depend on one, and only one, other subset of components, result in fan-like graphical models. (b) These graphical models can be visually compressed by packing the dependent nodes into a single plate.

This plate notation is especially useful when we have an arbitrary number of component spaces whose behavior conditionally depends on the same component spaces. For instance, we might

have the product space $X = X^{1:I} \times X_{I+1}$ and a joint distribution built up from the conditional decomposition

$$p(x_1, \ldots, x_{I+1}) = \left[\prod_{i=1}^{I} p(x_i \mid x_{I+1}) \right] p(x_{I+1}).$$

We can try to construct heuristic visualizations to represent the arbitrary number of nodes (Figure 15a), but the plate notation allow us to visualize this structure exactly (Figure 15b).



(a) (b)

Figure 15: (a) Graphical models with an arbitrary number of nodes are awkward to visualize. (b) When applicable, plates avoid this awkwardness entirely.

Plates can also be nested to accommodate *hierarchical conditional dependencies*

4.5 Subtleties and Limitations

Directed graphical models can be a powerful tool for communicating the structure of conditional decompositions in practice, especially as a complement to explicit probability density functions. That said, directed graphical models are without their subtleties and limitations that require some care.

For instance, directed graphical models communicate the structure of a particular *conditional decomposition*, not a particular *joint distribution*. The same joint distribution under different conditional decompositions will result in different graphical models. At the same time, different joint distributions with the same conditional dependencies will correspond to the same graphical model, just with differences in the conditional distributions and/or marginal distribution.

In other words, graphical models does no completely specify joint distributions. Instead it defines a kind of mathematical "armature" around which we can build up a joint distribution with the choice of particular conditional and marginal probability distributions. In order

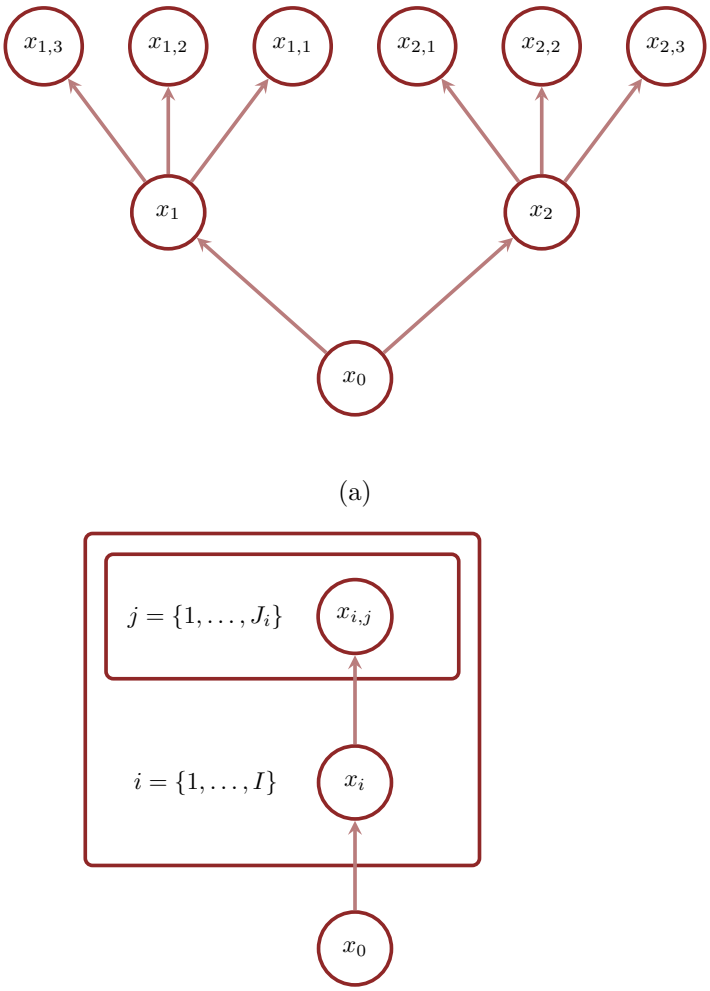


Figure 16: (a) Nested hierarchies of conditional dependencies can also be compactly visualized by (b) nested plates within other plates.

to communicate those choices, we need to complement a graphical model with additional information.

Another potential issue is that transforming a joint distribution will, in general, transform the corresponding graphical model. Transformations that preserve the product structure of a space will, by definition, also preserve the structure of conditional decompositions, and hence the structure of of graphical models. For instance, we can freely transform individual component spaces without affecting the graphical modeling representation. On the other hand, transformations that reorder the component spaces or mix them together will usually change the conditional decomposition, and hence the corresponding graphical model.

One of the more frustrating aspects of directed graphical models is that they can struggle to effectively communicate certain operations.

Consider, for example, the joint distribution defined by the density function decomposition

$$\begin{aligned} p(x_1, \ldots, x_6) = \; & p(x_1 \mid x_4 - x_6) \\ & \cdot p(x_2 \mid x_6 - x_5) \\ & \cdot p(x_3 \mid x_5 - x_4) \\ & \cdot p(x_4)\, p(x_5)\, p(x_6). \end{aligned}$$

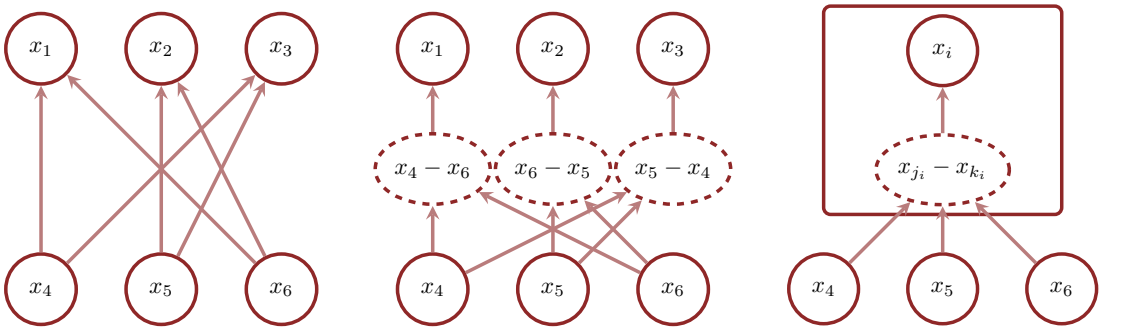
A graphical model using only six nodes, one for each X_i, can be confused with a more general conditional dependencies (Figure 17a),

$$\begin{aligned} p(x_1, \ldots, x_6) = \; & p(x_1 \mid x_4, x_6) \\ & \cdot p(x_2 \mid x_5, x_6) \\ & \cdot p(x_3 \mid x_4, x_5) \\ & \cdot p(x_4)\, p(x_5)\, p(x_6). \end{aligned}$$

We can capture the conditional dependence on the differences with pushforward nodes (Figure 17a), but the overlapping arrows can be difficult to parse. Moreover, the parsing becomes only more difficult as we add more nodes that also depend on differences between x_4, x_5, and x_6.

Another possibility is to employ plate notation (Figure 17c). The problem with this approach, however, is that each y_i doesn't depend on x_4, x_5, and x_6 all at the same time. Consequently, this plate diagram is fundamentally ambiguous as well.

All of this is to say that graphical models are not always the best tool for communicating conditional dependencies. Sometimes a probability density function decomposition is just more effective. Other times we might need to reach for other representations, such as *probabilistic programming*. This will be a topic for a future chapter.



$p(x_1, x_2, x_3, x_4, x_5, x_6)$
$= p(x_1 \mid x_4, x_6)$
$\cdot p(x_2 \mid x_5, x_6)$
$\cdot p(x_3 \mid x_4, x_5)$
$\cdot p(x_4)\, p(x_5)\, p(x_6)$

(a)

$p(x_1, x_2, x_3, x_4, x_5, x_6)$
$= p(x_1 \mid x_4 - x_6)$
$\cdot p(x_2 \mid x_6 - x_5)$
$\cdot p(x_3 \mid x_5 - x_4)$
$\cdot p(x_4)\, p(x_5)\, p(x_6)$

(b)

$p(x_1, x_2, x_3, x_4, x_5, x_6)$
$= \prod_{i=1}^{3} p(x_i \mid x_{j_i} - x_{k_i})$
$\cdot p(x_4)\, p(x_5)\, p(x_6)$

(c)

Figure 17: Some conditional decompositions are just awkward to to represent with graphical models. Consider, for example, a joint density function built up from the conditional density functions $p(x_1 \mid x_4 - x_6)$, $p(x_2 \mid x_6 - x_5)$, and $p(x_3 \mid x_5 - x_4)$. (a) The immediate graphical model doesn't communicate the particular dependence on differences. (b) Introducing pushforward nodes helps, but at the expense of a much busier graphical model. (c) Using plates results in a more elegant graphical model, but requires introducing conditional indices j_i and k_i whose behavior is left ambiguous.

5 Bayes' Theorem

Given a subset of components, we can decompose a joint distribution two different ways. The conditional probability kernel and a tailing marginal distribution from each of these decompositions, however, are intimately related.

Consider the complementary component subsets

$$\mathsf{i} \subset \{1, \ldots, I\}$$

and

$$\mathsf{i}^c \subset \{1, \ldots, I\}.$$

The partition

$$(\mathsf{i}^c, \mathsf{i})$$

defines the conditional probability kernel

$$\pi^{\varpi_{\mathsf{i}}}(\mathsf{x}_{\mathsf{i}^c} \mid x_{\mathsf{i}})$$

and the marginal probability distribution

$$(\varpi_{\mathsf{i}})_* \pi.$$

At the same time, the permuted partition

$$(\mathsf{i}, \mathsf{i}^c)$$

defines the conditional probability kernel

$$\pi^{\varpi_{\mathsf{i}^c}}(\mathsf{x}_{\mathsf{i}} \mid x_{\mathsf{i}^c})$$

and the marginal probability distribution

$$(\varpi_{\mathsf{i}^c})_* \pi.$$

Each conditional probability distribution in the conditional probability kernel of the *first* decomposition is defined on a cross section,

$$X^{\mathsf{i}^c} \times \{x_{\mathsf{i}}\},$$

For fixed x_{i}, this cross section is equivalent to the product space

$$X^{\mathsf{i}^c}.$$

This, however, is exactly the space over which the marginal probability distribution in the *second* decomposition is defined. Consequently, we can define Radon-Nikodym derivatives between these objects,

$$\frac{\mathrm{d}\pi^{\varpi_{\mathsf{i}}}}{\mathrm{d}(\varpi_{\mathsf{i}^c})_* \pi}(x_{\mathsf{i}^c} \mid x_{\mathsf{i}}).$$

Similarly, we can construct Radon-Nikodym derivatives between each conditional probability distribution in the *second* decomposition and the marginal probability distribution in the first decomposition,

$$\frac{\mathrm{d}\pi^{\varpi_{\mathsf{i}^c}}}{\mathrm{d}(\varpi_{\mathsf{i}})_*\pi}(x_{\mathsf{i}} \mid x_{\mathsf{i}^c}).$$

In order for the two decompositions to define the same joint probability distribution, these Radon-Nikodym derivatives have to be almost always equal,

$$\frac{\mathrm{d}\pi^{\varpi_{\mathsf{i}}}}{\mathrm{d}(\varpi_{\mathsf{i}^c})_*\pi}(x_{\mathsf{i}^c} \mid x_{\mathsf{i}}) \overset{\pi}{=} \frac{\mathrm{d}\pi^{\varpi_{\mathsf{i}^c}}}{\mathrm{d}(\varpi_{\mathsf{i}})_*\pi}(x_{\mathsf{i}} \mid x_{\mathsf{i}^c}).$$

The identification between these Radon-Nikodym derivatives admits a particularly compelling interpretation. Consider the conditional expectation

$$\int \pi^{\varpi_{\mathsf{i}^c}}(\mathrm{d}x_{\mathsf{i}} \mid \tilde{x}_{\mathsf{i}^c})\, g(x_{\mathsf{i}}).$$

Using Radon-Nikodym derivatives, we can compute this conditional expectation as a complementary marginal expectation,

$$\int \pi^{\varpi_{\mathsf{i}^c}}(\mathrm{d}_{\mathsf{i}} \mid \tilde{x}_{\mathsf{i}^c})\, g(x_{\mathsf{i}}) = \int (\varpi_{\mathsf{i}})_*\pi(\mathrm{d}x_{\mathsf{i}})\, \frac{\mathrm{d}\pi^{\varpi_{\mathsf{i}^c}}}{\mathrm{d}(\varpi_{\mathsf{i}})_*\pi}(x_{\mathsf{i}} \mid \tilde{x}_{\mathsf{i}^c})\, g(x_{\mathsf{i}}).$$

With the above identity, we can swap the Radon-Nikodym derivative to give

$$\int \pi^{\varpi_{\mathsf{i}^c}}(\mathrm{d}_{\mathsf{i}} \mid \tilde{x}_{\mathsf{i}^c})\, g(x_{\mathsf{i}}) = \int (\varpi_{\mathsf{i}})_*\pi(\mathrm{d}x_{\mathsf{i}})\, \frac{\mathrm{d}\pi^{\varpi_{\mathsf{i}}}}{\mathrm{d}(\varpi_{\mathsf{i}^c})_*\pi}(\tilde{x}_{\mathsf{i}^c} \mid x_{\mathsf{i}})\, g(x_{\mathsf{i}}).$$

Comparing this to the parallel marginal expectation,

$$\int (\varpi_{\mathsf{i}})_*\pi(\mathrm{d}x_{\mathsf{i}})\, g(x_{\mathsf{i}}),$$

we can "update" a marginal expectation into a conditional expectation, where $\tilde{x}_{\mathsf{i}^c}$ is fixed, by scaling the expectand with the Radon-Nikodym derivative

$$\frac{\mathrm{d}\pi^{\varpi_{\mathsf{i}}}}{\mathrm{d}(\varpi_{\mathsf{i}^c})_*\pi}(\tilde{x}_{\mathsf{i}^c} \mid x_{\mathsf{i}}).$$

Equivalently,

$$\pi^{\varpi_{\mathsf{i}^c}}(\mathrm{d}_{\mathsf{i}} \mid \tilde{x}_{\mathsf{i}^c}) = \frac{\mathrm{d}\pi^{\varpi_{\mathsf{i}}}}{\mathrm{d}(\varpi_{\mathsf{i}^c})_*\pi}(\tilde{x}_{\mathsf{i}^c} \mid x_{\mathsf{i}})\, (\varpi_{\mathsf{i}})_*\pi(\mathrm{d}x_{\mathsf{i}}).$$

This relationship between marginal and conditional expectation values is known as **Bayes' Theorem**. The Reverend Thomas Bayes first published a special case of this relationship all the way back in 1763 (Bayes 1763), albeit posthumously.

This general form of Bayes' Theorem is, admittedly, a bit abstract. Fortunately, it becomes much more straightforward in terms of probability density functions. Assuming consistent reference measures, the component subset i induces the probability density function decompositions

$$p(x_{\mathsf{i}}, x_{\mathsf{i}^c}) \overset{\nu^{1:I}}{=} p(x_{\mathsf{i}} \mid x_{\mathsf{i}^c})\, p(x_{\mathsf{i}^c})$$

and

$$p(x_{\mathsf{i}}, x_{\mathsf{i}^c}) \overset{\nu^{1:I}}{=} p(x_{\mathsf{i}^c} \mid x_{\mathsf{i}})\, p(x_{\mathsf{i}}).$$

Bayes' Theorem relates all of these density functions together,

$$\begin{aligned}
p(x_{\mathsf{i}} \mid x_{\mathsf{i}^c}) &\overset{\nu^{1:I}}{=} \frac{p(x_{\mathsf{i}^c}, x_{\mathsf{i}})}{p(x_{\mathsf{i}^c})}\, p(x_{\mathsf{i}}) \\
&\overset{\nu^{1:I}}{=} \frac{p(x_{\mathsf{i}^c} \mid x_{\mathsf{i}})\, p(x_{\mathsf{i}})}{p(x_{\mathsf{i}^c})} \\
&\overset{\nu^{1:I}}{=} \frac{p(x_{\mathsf{i}^c} \mid x_{\mathsf{i}}))}{p(x_{\mathsf{i}^c})}\, p(x_{\mathsf{i}}) \\
&\overset{\nu^{1:I}}{=} \frac{p(x_{\mathsf{i}^c} \mid x_{\mathsf{i}})}{\int \nu^{\mathsf{i}}(\mathrm{d}x_{\mathsf{i}})\, p(x_{\mathsf{i}}, x_{\mathsf{i}^c}).}\, p(x_{\mathsf{i}}).
\end{aligned}$$

Conveniently, we can quickly derive this density version of Bayes' Theorem in a few different ways. For example, the identification of the mixed Radon-Nikodym derivatives implies

$$\begin{aligned}
\frac{\mathrm{d}\pi^{\varpi_{\mathsf{i}}}}{\mathrm{d}(\varpi_{\mathsf{i}^c})_* \pi}(x_{\mathsf{i}^c} \mid x_{\mathsf{i}}) &\overset{\nu^{1:I}}{=} \frac{\mathrm{d}\pi^{\varpi_{\mathsf{i}^c}}}{\mathrm{d}(\varpi_{\mathsf{i}})_* \pi}(x_{\mathsf{i}} \mid x_{\mathsf{i}^c}) \\
\frac{p(x_{\mathsf{i}^c} \mid x_{\mathsf{i}})}{p(x_{\mathsf{i}^c})} &\overset{\nu^{1:I}}{=} \frac{p(x_{\mathsf{i}} \mid x_{\mathsf{i}^c})}{p(x_{\mathsf{i}})},
\end{aligned}$$

which we can immediately manipulate into Bayes' Theorem,

$$\begin{aligned}
\frac{p(x_{\mathsf{i}^c} \mid x_{\mathsf{i}})}{p(x_{\mathsf{i}^c})} &\overset{\nu^{1:I}}{=} \frac{p(x_{\mathsf{i}} \mid x_{\mathsf{i}^c})}{p(x_{\mathsf{i}})} \\
p(x_{\mathsf{i}} \mid x_{\mathsf{i}^c}) &\overset{\nu^{1:I}}{=} \frac{p(x_{\mathsf{i}^c} \mid x_{\mathsf{i}})}{p(x_{\mathsf{i}^c})}\, p(x_{\mathsf{i}}.
\end{aligned}$$

Perhaps the easiest way to derive Bayes' Theorem, however, is to start with the consistency of the conditional decompositions directly,

$$\begin{aligned}
p(x_{\mathsf{i}}, x_{\mathsf{i}^c}) &\overset{\nu^{1:I}}{=} p(x_{\mathsf{i}}, x_{\mathsf{i}^c}) \\
p(x_{\mathsf{i}} \mid x_{\mathsf{i}^c})\, p(x_{\mathsf{i}^c}) &\overset{\nu^{1:I}}{=} p(x_{\mathsf{i}^c} \mid x_{\mathsf{i}})\, p(x_{\mathsf{i}}) \\
p(x_{\mathsf{i}} \mid x_{\mathsf{i}^c}) &\overset{\nu^{1:I}}{=} \frac{p(x_{\mathsf{i}^c} \mid x_{\mathsf{i}})}{p(x_{\mathsf{i}^c})}\, p(x_{\mathsf{i}}).
\end{aligned}$$

Finally, Bayes' Theorem becomes somewhat trivial to implement when we need only an single unnormalized conditional density function. As we saw in **Section Blah**, the unnormalized conditional density function is immediately given by partially evaluating the joint density function,

$$\begin{aligned} p(x_{\mathsf{i}}, \tilde{x}_{\mathsf{i}^c}) &\overset{\nu^{1:I}}{=} \frac{p(\tilde{x}_{\mathsf{i}^c} \mid x_{\mathsf{i}})}{p(\tilde{x}_{\mathsf{i}^c})} \, p(x_{\mathsf{i}}) \\ &\overset{\nu^{1:I}}{\propto} p(\tilde{x}_{\mathsf{i}^c} \mid x_{\mathsf{i}}) \, p(x_{\mathsf{i}}) \\ &\overset{\nu^{1:I}}{\propto} p(\tilde{x}_{\mathsf{i}^c}, x_{\mathsf{i}}). \end{aligned}$$

Indeed, this is how Bayes' Theorem is most often implemented in practice.

6 Convolution

All of this product space probability theory can also be applied to probabilistic systems that don't necessarily have any product structure of their own.

Consider, for example, the probability space

$$(X, \mathcal{X}, \pi).$$

Even if the ambient space X does not have any product structure, we can always introduce product structure by combining it with itself. Specifically, we can construct the power space

$$X^2 = X \times X'$$

which is naturally equipped with the projection functions

$$\begin{aligned} \varpi : X^2 &\to X \\ (x, x') &\mapsto x \, , \end{aligned}$$

and

$$\begin{aligned} \varpi' : X^2 &\to X \\ (x, x') &\mapsto x'. \end{aligned}$$

Once we have defined X^2 and its projection functions, we can introduce a conditional probability kernel over the level sets of ϖ,

$$\pi(\mathrm{d}x' \mid x).$$

Any choice of conditional probability kernel lifts the initial probability distribution π to a joint probability distribution over the power space,

$$\pi(\mathrm{d}x' \mid x) \, \pi(\mathrm{d}x) = \pi(\mathrm{d}x, \mathrm{d}x').$$

By construction, pushing this joint distribution forward along ϖ returns the original probability distribution,

$$(\varpi)_* \pi(\mathrm{d}x) = \pi(\mathrm{d}x).$$

Pushing it forward along the *second* projection function ϖ', however, defines an entirely new probability distribution over X,

$$(\varpi')_* \pi(\mathrm{d}x') = \int \pi(\mathrm{d}x' \mid x)\, \pi(\mathrm{d}x) \neq \pi(\mathrm{d}x').$$

This process of lifting and then pushing forward along the converse projection function, transforming an initial probability distribution into a new probability distribution, is known as **convolution**. The conditional probability kernel defined by $p(x' \mid x)$ is referred to as a **convolution kernel**.

Consider, for example, the integers $X = \mathbb{I}$ equipped with a probability distribution defined by the Poisson density function

$$p(x) = \text{Poisson}(x \mid \lambda) = \frac{\lambda^x e^{-\lambda}}{x!}.$$

and the somewhat awkward convolution kernel defined by the conditional density functions

$$p(x' \mid x) = \frac{\eta^{x'-x} e^{-\eta}}{(x'-x)!} I[x \leq x'].$$

The convolution of this initial probability distribution with the convolution kernel is a new probability distribution. After some algebra, which is worked out in Appendix A.2, the new probability distribution is defined by another Poisson density function,

$$\begin{aligned} p(x') &= \sum_{x=0}^{\infty} p(x' \mid x)\, p(x) \\ &= \text{Poisson}(x' \mid \lambda + \eta). \end{aligned}$$

In this case, the convolution translates the intensity of the Poisson parameter by η.

For a second demonstration, consider a rigid real line $X = \mathbb{R}$ equipped with a probability distribution defined by the normal density function

$$\begin{aligned} p(x) &= \text{normal}(x \mid \mu, \sigma) \\ &= \frac{1}{\sqrt{2\,\pi\,\sigma^2}} \exp\left[-\frac{1}{2} \frac{(x-\mu)^2}{\sigma^2} \right]. \end{aligned}$$

Given a convolution kernel defined by the conditional density functions

$$\begin{aligned} p(x' \mid x) &= \text{normal}(x' \mid \eta - x, \tau) \\ &= \frac{1}{\sqrt{2\,\pi\,\tau^2}} \exp\left[-\frac{1}{2}\frac{(x' - \eta + x)^2}{\tau^2}\right], \end{aligned}$$

we can construct the convolution

$$\begin{aligned} p(x') &= \int \mathrm{d}x\, p(x, x') \\ &= \text{normal}\left(x' \mid \mu + \eta, \sqrt{\sigma^2 + \tau^2}\right). \end{aligned}$$

The full derivation can be found in Appendix A.3.

In this case, the convolution operation takes in initial normal density function, translates the location parameter by η, and then inflates the square of the scale.

When we take the limit $\tau \to 0$, the convolution becomes a pure translation of the location parameter,

$$p(x') = \text{normal}\left(x' \mid \mu + \eta, \sqrt{\sigma^2 + \tau^2}\right).$$

Indeed, this is exactly the pushforward of a normal density function along the translation operation that we discussed in Chapter 7, Section 4.3.2.1.

The convolution kernel in this limit acts like a singular delta function that shifts an input x to an output $x + \eta$. This limiting convolution is also known as a **linear convolution**.

7 Conclusion

All of the abstraction of conditional probability theory becomes concrete when we apply it to product spaces. Fortunately, all of the applications that we will consider in this book will be on product spaces. Consequently, the discussion in this chapter will be the foundation all of the work we will do moving forwards.

Appendix: "Explicit" Calculations

Per tradition, I have have sequestered the more involved calculations in this chapter to this appendix.

A.1 Density Function Disintegration

Consider the binary product space $X^{1:2} = \mathbb{R}^2$ equipped with a two-dimensional Lebesgue measure and the joint distribution defined by the joint density function

$$p(x_1, x_2) = \frac{1}{2\,\pi\,\sqrt{\sigma_1^2\,\sigma_2^2\,(1-\rho^2)}} \exp\left[-\frac{1}{2} Q(x_1, x_2)\right]$$

where

$$Q(x_1, x_2) = \frac{\sigma_2^2\,x_1^2 - 2\,\rho\,\sigma_1\,\sigma_2\,x_1\,x_2 + \sigma_1^2\,x_2^2}{\sigma_1^2\,\sigma_2^2\,(1-\rho^2).}$$

The marginal density function over X_1 is given by integrating over x_2,

$$\begin{aligned} p(x_1) &= \int \mathrm{d}x_2\, p(x_1, x_2) \\ &= \frac{1}{2\,\pi\,\sqrt{\sigma_1^2\,\sigma_2^2\,(1-\rho^2)}} \int \mathrm{d}x_2\, \exp\left[-\frac{1}{2} Q(x_1, x_2)\right]. \end{aligned}$$

In order to evaluate this integral, we need to isolate all of the x_2 terms in $Q(x_1, x_2)$. This requires completing the square for x_2 in the numerator,

$$\begin{aligned} \sigma_2^2\,x_1^2 - 2\,\rho\,\sigma_1\,\sigma_2\,x_1\,x_2 + \sigma_1^2\,x_2^2 &= \sigma_1^2\,\left(x_2 - \rho\frac{\sigma_2}{\sigma_1} x_1\right)^2 + \sigma_2^2\,x_1^2 - \rho\,\sigma_2^2\,x_1^2 \\ &= \sigma_1^2\,\left(x_2 - \rho\frac{\sigma_2}{\sigma_1} x_1\right)^2 + \sigma_2^2\,x_1^2\,(1-\rho^2). \end{aligned}$$

Then

$$\begin{aligned} Q(x_1, x_2) &= \frac{\sigma_2^2\,x_1^2 - 2\,\rho\,\sigma_1\,\sigma_2\,x_1\,x_2 + \sigma_1^2\,x_2^2}{\sigma_1^2\,\sigma_2^2\,(1-\rho^2)} \\ &= \frac{\sigma_1^2\,\left(x_2 - \rho\frac{\sigma_2}{\sigma_1} x_1\right)^2 + \sigma_2^2\,x_1^2\,(1-\rho^2)}{\sigma_1^2\,\sigma_2^2\,(1-\rho^2)} \\ &= \frac{\left(x_2 - \rho\frac{\sigma_2}{\sigma_1} x_1\right)^2}{\sigma_2^2\,(1-\rho^2)} + \frac{x_1^2}{\sigma_1^2}. \end{aligned}$$

This allows us to write the joint density function as

$$\begin{aligned}
p(x_1, x_2) &= \frac{1}{2\,\pi\,\sqrt{\sigma_1^2\,\sigma_2^2\,(1-\rho^2)}}\,\exp\left[-\frac{1}{2}Q(x_1, x_2)\right] \\
&= \frac{1}{2\,\pi\,\sqrt{\sigma_1^2\,\sigma_2^2\,(1-\rho^2)}}\,\exp\left[-\frac{1}{2}\left(\frac{\left(x_2 - \rho\frac{\sigma_2}{\sigma_1}x_1\right)^2}{\sigma_2^2\,(1-\rho^2)} + \frac{x_1^2}{\sigma_1^2}\right)\right] \\
&= \frac{1}{2\,\pi\,\sqrt{\sigma_1^2\,\sigma_2^2\,(1-\rho^2)}}\,\exp\left[-\frac{1}{2}\frac{\left(x_2 - \rho\frac{\sigma_2}{\sigma_1}x_1\right)^2}{\sigma_2^2\,(1-\rho^2)}\right]\exp\left[-\frac{1}{2}\frac{x_1^2}{\sigma_1^2}\right].
\end{aligned}$$

With this factorization, the marginal density function reduces to a normal integral which we can evaluate in closed form,

$$\begin{aligned}
p(x_1) &= \int \mathrm{d}x_2\,p(x_1, x_2) \\
&= \frac{1}{2\,\pi\,\sqrt{\sigma_1^2\,\sigma_2^2\,(1-\rho^2)}}\int \mathrm{d}x_2\,\exp\left[-\frac{1}{2}\frac{\left(x_2 - \rho\frac{\sigma_2}{\sigma_1}x_1\right)^2}{\sigma_2^2\,(1-\rho^2)}\right]\exp\left[-\frac{1}{2}\frac{x_1^2}{\sigma_1^2}\right] \\
&= \frac{1}{2\,\pi\,\sqrt{\sigma_1^2\,\sigma_2^2\,(1-\rho^2)}}\,\exp\left[-\frac{1}{2}\frac{x_1^2}{\sigma_1^2}\right]\int \mathrm{d}x_2\,\exp\left[-\frac{1}{2}\frac{\left(x_2 - \rho\frac{\sigma_2}{\sigma_1}x_1\right)^2}{\sigma_2^2\,(1-\rho^2)}\right] \\
&= \frac{1}{2\,\pi\,\sqrt{\sigma_1^2\,\sigma_2^2\,(1-\rho^2)}}\,\exp\left[-\frac{1}{2}\frac{x_1^2}{\sigma_1^2}\right]\sqrt{2\,\pi\,\sigma_2^2\,(1-\rho^2)} \\
&= \frac{1}{\sqrt{2\,\pi\,\sigma_1^2}}\,\exp\left[-\frac{1}{2}\frac{x_1^2}{\sigma_1^2}\right].
\end{aligned}$$

After all of the that, the marginal density function is just a normal density function,

$$\begin{aligned}
p(x_1) &= \frac{1}{\sqrt{2\,\pi\,\sigma_1^2}}\,\exp\left[-\frac{1}{2}\frac{x_1^2}{\sigma_1^2}\right] \\
&= \mathrm{normal}(x_1 \mid 0, \sigma_1).
\end{aligned}$$

We can use this same factorization of the joint density function to simplify the calculation of

the conditional density functions,

$$\begin{aligned}
p(x_2 \mid x_1) &= \frac{p(x_1, x_2)}{p(x_1)} \\
&= \frac{\frac{1}{2\,\pi\,\sqrt{\sigma_1^2\,\sigma_2^2\,(1-\rho^2)}} \exp\left[-\frac{1}{2} \frac{\left(x_2 - \rho\frac{\sigma_2}{\sigma_1} x_1\right)^2}{\sigma_2^2\,(1-\rho^2)} \right] \exp\left[-\frac{1}{2}\frac{x_1^2}{\sigma_1^2} \right]}{\frac{1}{\sqrt{2\,\pi\,\sigma_1^2}} \exp\left[-\frac{1}{2}\frac{x_1^2}{\sigma_1^2} \right]} \\
&= \frac{1}{\sqrt{2\,\pi\,\sigma_2^2\,(1-\rho^2)}} \exp\left[-\frac{1}{2} \frac{\left(x_2 - \rho\frac{\sigma_2}{\sigma_1} x_1\right)^2}{\sigma_2^2\,(1-\rho^2)} \right].
\end{aligned}$$

Perhaps surprisingly, each conditional density function is also a normal density function,

$$\begin{aligned}
p(x_2 \mid x_1) &= \frac{1}{\sqrt{2\,\pi\,\sigma_2^2\,(1-\rho^2)}} \exp\left[-\frac{1}{2} \frac{\left(x_2 - \rho\frac{\sigma_2}{\sigma_1} x_1\right)^2}{\sigma_2^2\,(1-\rho^2)} \right] \\
&= \text{normal}\left(x_2 \,\middle|\, \rho\frac{\sigma_2}{\sigma_1} x_1, \sigma_2\sqrt{1-\rho^2} \right).
\end{aligned}$$

Because of the symmetry of the joint density function, the same calculations apply if when deriving the other marginal density function and corresponding conditional density functions. The result just swaps all of the indices,

$$\begin{aligned}
p(x_2) &= \text{normal}(x_2 \mid 0, \sigma_2) \\
p(x_1 \mid x_2) &= \text{normal}\left(x_1 \,\middle|\, \rho\frac{\sigma_1}{\sigma_2} x_2, \sigma_1\sqrt{1-\rho^2} \right).
\end{aligned}$$

Finally, this is an example where some pattern matching can point to the marginal and conditional density functions directly without having to evaluate a single integral. Examining

$$p(x_1, x_2) = \frac{1}{2\,\pi\,\sqrt{\sigma_1^2\,\sigma_2^2\,(1-\rho^2)}} \exp\left[-\frac{1}{2} \frac{\left(x_2 - \rho\frac{\sigma_2}{\sigma_1} x_1\right)^2}{\sigma_2^2\,(1-\rho^2)} \right] \exp\left[-\frac{1}{2}\frac{x_1^2}{\sigma_1^2} \right],$$

we might notice that both of the exponentiated quadratics define normal density functions,

$$\begin{aligned}
p(x_1, x_2) &= \frac{1}{2\,\pi\,\sqrt{\sigma_1^2\,\sigma_2^2\,(1-\rho^2)}} \exp\left[-\frac{1}{2} \frac{\left(x_2 - \rho\frac{\sigma_2}{\sigma_1} x_1\right)^2}{\sigma_2^2\,(1-\rho^2)} \right] \exp\left[-\frac{1}{2}\frac{x_1^2}{\sigma_1^2} \right] \\
&= \frac{1}{\sqrt{2\,\pi\,\sigma_2^2\,(1-\rho^2)}} \exp\left[-\frac{1}{2} \frac{\left(x_2 - \rho\frac{\sigma_2}{\sigma_1} x_1\right)^2}{\sigma_2^2\,(1-\rho^2)} \right] \frac{1}{\sqrt{2\,\pi\,\sigma_1^2}} \exp\left[-\frac{1}{2}\frac{x_1^2}{\sigma_1^2} \right] \\
&= \text{normal}\left(x_1 \,\middle|\, \rho\frac{\sigma_1}{\sigma_2} x_2, \sigma_1\sqrt{1-\rho^2} \right) \text{normal}(x_1 \mid 0, \sigma_1).
\end{aligned}$$

In other words, by isolating x_2 we have incidentally manipulated the joint density function into the desired conditional decomposition

$$p(x_1, x_2) = p(x_2 \mid x_1)\, p(x_1).$$

This approach isn't always useful, in particular it requires being able to identify normalized density functions by eye. That said, it can save time when working with similar calculations.

A.2 Discrete Convolution

We begin with the space $X = \mathbb{I}$ equipped with a counting reference measure and a probability distribution defined by the Poisson density function

$$p(x) = \text{Poisson}(x \mid \lambda) = \frac{\lambda^x e^{-\lambda}}{x!}.$$

Here we'll consider the convolution kernel defined by the conditional density functions

$$p(x' \mid x) = \frac{\eta^{x'-x} e^{-\eta}}{(x'-x)!} I[x \leq x'].$$

Before deriving the convolution, however, let's first verify that this somewhat awkward looking object is, in fact, a conditional probability kernel. To do that, we'll need to verify that for each conditional distribution is properly normalized for all values of $\eta > 0$.

The normalizations are given by explicit summations,

$$\begin{aligned}
\sum_{x'=0}^{\infty} p(x' \mid x) &= \sum_{x'=0}^{\infty} \frac{\eta^{x'-x} e^{-\eta}}{(x'-x)!} I[x \leq x'] \\
&= \sum_{x'=x}^{\infty} \frac{\eta^{x'-x} e^{-\eta}}{(x'-x)!}.
\end{aligned}$$

Changing the index of the summation to $k = x' - x$ gives

$$\begin{aligned}
\sum_{x'=0}^{\infty} p(x' \mid x) &= \sum_{k=0}^{\infty} \frac{\eta^k e^{-\eta}}{k!} \\
&= e^{-\eta} \sum_{k=0}^{\infty} \frac{\eta^k}{k!} \\
&= e^{-\eta} e^{\eta} \\
&= 1,
\end{aligned}$$

as required.

With the validity of the convolution kernel verified, we can proceed to the convolution itself,

$$\begin{aligned} p(x') &= \sum_{x=0}^{\infty} p(x' \mid x)\, p(x) \\ &= \sum_{x=0}^{\infty} \frac{\eta^{x'-x} e^{-\eta}}{(x'-x)!} I[x \leq x'] \, \frac{\lambda^x e^{-\lambda}}{x!} \\ &= e^{-(\lambda+\eta)} \sum_{x=0}^{\infty} I[x \leq x'] \, \frac{1}{x!\,(x'-x)!} \eta^{x'-x}\, \lambda^x \\ &= \frac{e^{-(\lambda+\eta)}}{x'!} \sum_{x=0}^{x'} \frac{x'!}{x!\,(x'-x)!} \eta^{x'-x}\, \lambda^x. \end{aligned}$$

Conveniently, the summation is immediately given by the binomial theorem,

$$(a+b)^K = \sum_{k=0}^{K} a^k\, b^{K-k}.$$

Consequently,

$$\begin{aligned} p(x') &= \frac{e^{-(\lambda+\eta)}}{x'!} \sum_{x=0}^{x'} \frac{x'!}{x!\,(x'-x)!} \eta^{x'-x}\, \lambda^x \\ &= \frac{e^{-(\lambda+\eta)}}{x'!} (\lambda+\eta)^{x'}. \end{aligned}$$

This, however, is just another Poisson density function,

$$p(x') = \frac{e^{-(\lambda+\eta)}}{x'!} (\lambda+\eta)^{x'} = \text{Poisson}(x' \mid \lambda+\eta).$$

A.3 Continuous Convolution

Consider a rigid real line $X = \mathbb{R}$ equipped with a Lebesgue reference measure and a probability distribution defined by the normal density function

$$\begin{aligned} p(x) &= \text{normal}(x \mid \mu, \sigma) \\ &= \frac{1}{\sqrt{2\,\pi\,\sigma^2}} \exp\left[-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right]. \end{aligned}$$

The convolution kernel defined by the conditional density functions

$$\begin{aligned} p(x' \mid x) &= \text{normal}(x' \mid \eta - x, \tau) \\ &= \frac{1}{\sqrt{2\,\pi\,\tau^2}} \exp\left[-\frac{1}{2}\frac{(x'-\eta+x)^2}{\tau^2}\right]. \end{aligned}$$

lifts the initial probability distribution into a joint distribution over X^2 defined by the joint density function

$$\begin{aligned} p(x, x') &= p(x' \mid x)\, p(x) \\ &= \text{normal}(x \mid \mu, \sigma)\, \text{normal}(x' \mid \eta - x, \tau) \\ &= \frac{1}{2\,\pi\,\sqrt{\sigma^2\,\tau^2}} \exp\left[-\frac{1}{2} Q(x, x')\right], \end{aligned}$$

where

$$Q(x, x') = \frac{(x - \mu)^2}{\sigma^2} + \frac{(x' - \eta + x)^2}{\tau^2}.$$

The convolution of the initial probability distribution with the convolution kernel is given by projecting this joint density function to the auxiliary space,

$$\begin{aligned} p(x') &= \int \mathrm{d}x\, p(x, x') \\ &= \frac{1}{2\,\pi\,\sqrt{\sigma^2\,\tau^2}} \int \mathrm{d}x\, \exp\left[-\frac{1}{2} Q(x, x')\right]. \end{aligned}$$

To evaluate this integral, we'll need to isolate x with some – surprise! – completion of squares,

$$
\begin{aligned}
Q(x,x') &= \frac{(x-\mu)^2}{\sigma^2} + \frac{(x'-\eta+x)^2}{\tau^2} \\
&= \frac{x^2 - 2\,\mu\,x + \mu^2}{\sigma^2} + \frac{x^2 - 2\,(x'-\eta)\,x + (x'-\eta)^2}{\tau^2} \\
&= \frac{1}{\sigma^2\,\tau^2}\left[(\sigma^2+\tau^2)\,x^2 - 2(\tau^2\,\mu + \sigma^2\,(x'-\eta))\,x + \tau^2\,\mu^2 + \sigma^2(x'-\eta)^2\right] \\
&= \frac{1}{\sigma^2\,\tau^2}\Bigg[\quad (\sigma^2+\tau^2)\,\left(x - \frac{\tau^2\,\mu + \sigma^2\,(x'-\eta)}{\sigma^2+\tau^2}\right)^2 \\
&\qquad\qquad + \tau^2\,\mu^2 + \sigma^2(x'-\eta)^2 - \frac{\left(\tau^2\,\mu + \sigma^2\,(x'-\eta)\right)^2}{\sigma^2+\tau^2}\Bigg] \\
&= \quad \frac{1}{\sigma^2\,\tau^2}\left[(\sigma^2+\tau^2)\,\left(x - \frac{\tau^2\,\mu + \sigma^2\,(x'-\eta)}{\sigma^2+\tau^2}\right)^2\right] \\
&\quad + \frac{1}{\sigma^2\,\tau^2}\Bigg[\quad \frac{\tau^4\,\mu^2 + \sigma^2\,\tau^2\,\mu^2 + \sigma^4(x'-\eta)^2 + \sigma^2\,\tau^2\,(x'-\eta)^2}{\sigma^2+\tau^2} \\
&\qquad\qquad - \frac{\tau^4\,\mu^2 + 2\tau^2\,\sigma^2\,\mu\,(x'-\eta) + \sigma^4(x'-\eta)^2}{\sigma^2+\tau^2} \qquad\Bigg] \\
&= \quad \frac{1}{\sigma^2\,\tau^2}\left[(\sigma^2+\tau^2)\,\left(x - \frac{\tau^2\,\mu + \sigma^2\,(x'-\eta)}{\sigma^2+\tau^2}\right)^2\right] \\
&\quad + \frac{1}{\sigma^2\,\tau^2}\left[\frac{\sigma^2\,\tau^2\,\mu^2 - 2\,\tau^2\,\sigma^2\,\mu\,(x'-\eta) + \sigma^2\,\tau^2\,(x'-\eta)^2}{\sigma^2+\tau^2}\right] \\
&= \quad \frac{\sigma^2+\tau^2}{\sigma^2\,\tau^2}\left(x - \frac{\tau^2\,\mu + \sigma^2\,(x'-\eta)}{\sigma^2+\tau^2}\right)^2 \\
&\quad + \frac{\mu^2 - 2\,\mu\,(x'-\eta) + (x'-\eta)^2}{\sigma^2+\tau^2} \\
&= \quad \frac{\sigma^2+\tau^2}{\sigma^2\,\tau^2}\left(x - \frac{\tau^2\,\mu + \sigma^2\,(x'-\eta)}{\sigma^2+\tau^2}\right)^2 \\
&\quad + \frac{(x'-(\mu+\eta))^2}{\sigma^2+\tau^2}.
\end{aligned}
$$

This allows us to reduce the convolution operation to a normal integral,

$$
\begin{aligned}
p(x') &= \frac{1}{2\,\pi\,\sqrt{\sigma^2\,\tau^2}} \int \mathrm{d}x\, \exp\left[-\frac{1}{2} Q(x, x')\right] \\
&= \frac{1}{2\,\pi\,\sqrt{\sigma^2\,\tau^2}} \int \mathrm{d}x\ \exp\left[-\frac{1}{2}\frac{\sigma^2+\tau^2}{\sigma^2\,\tau^2}\left(x - \frac{\tau^2\,\mu + \sigma^2\,(x'-\eta)}{\sigma^2+\tau^2}\right)^2\right] \\
&\qquad\qquad \cdot \exp\left[-\frac{1}{2}\frac{(x'-(\mu+\eta))^2}{\sigma^2+\tau^2}\right] \\
&= \frac{1}{2\,\pi\,\sqrt{\sigma^2\,\tau^2}}\ \exp\left[-\frac{1}{2}\frac{(x'-(\mu+\eta))^2}{\sigma^2+\tau^2}\right] \\
&\qquad \int \mathrm{d}x\, \exp\left[-\frac{1}{2}\frac{\sigma^2+\tau^2}{\sigma^2\,\tau^2}\left(x - \frac{\tau^2\,\mu + \sigma^2\,(x'-\eta)}{\sigma^2+\tau^2}\right)^2\right] \\
&= \frac{1}{2\,\pi\,\sqrt{\sigma^2\,\tau^2}}\ \exp\left[-\frac{1}{2}\frac{(x'-(\mu+\eta))^2}{\sigma^2+\tau^2}\right] \sqrt{2\,\pi\,\frac{\sigma^2\,\tau^2}{\sigma^2+\tau^2}} \\
&= \frac{1}{\sqrt{2\,\pi\,(\sigma^2+\tau^2)}}\ \exp\left[-\frac{1}{2}\frac{(x'-(\mu+\eta))^2}{\sigma^2+\tau^2}\right] \\
&= \mathrm{normal}\left(x' \mid \mu+\eta, \sqrt{\sigma^2+\tau^2}\right).
\end{aligned}
$$

Acknowledgements

A very special thanks to everyone supporting me on Patreon: Alessandro Varacca, Alex D, Alexander Noll, Amit, Andrea Serafino, Andrew Mascioli, Andrew Rouillard, Andrés Castro Araújo, Ara Winter, Ari Holtzman, Austin Rochford, Aviv Keshet, Avraham Adler, Ben Matthews, Ben Swallow, Benoit Essiambre, boot, Brendan Galdo, Bryan Chang, Cameron Smith, Canaan Breiss, Cat Shark, Cathy Oliveri, Charles Naylor, Chase Dwelle, Chris Jones, Christina Van Heer, Christopher Mehrvarzi, Colin Carroll, Colin McAuliffe, Damien Mannion, dan mackinlay, Dan W Joyce, Dan Waxman, Dan Weitzenfeld, Daniel Hammarström, Danny Van Nest, David Burdelski, Dr. Jobo, Dr. Omri Har Shemesh, Dylan Maher, Dylan Spielman, Ebriand, Ed Cashin, Eric LaMotte, Erik Banek, Eugene O'Friel, Felipe González, Felipe Vaca, Fergus Chadwick, Francesco Corona, Geoff Rollins, Glenn Williams, Granville Matheson, Guilherme Marthe, Hamed Bastan-Hagh, haubur, Hector Munoz, Horace Guy, hs, Hugo Botha, Håkan Johansson, Ian Costley, idontgetoutmuch, Ignacio Vera, Ilaria Prosdocimi, iris pisscauldron, Isaac Vock, jacob pine, Jair Andrade, James C, James Hodgson, James Wade, Janek Berger, Jarrett Byrnes, Jason Pekos, Jason Wong, jd, Jeff Burnett, Jeff Dotson, Jeff Helzner, Jeffrey Erlich, Jerry Lin , Jesper Fischer Ehmsen, Jessica Graves, Joe Sloan, John Flournoy, Jonathan H. Morgan, Jonathon Vallejo, Josh Knecht, JU, Julian Lee, Justin Bois, Karim Naguib, Karim Osman, Karsten Skogsholm, Konstantin Shakhbazov, Kristian Gårdhus Wichmann, Kádár András, Lars Barquist, lizzie , LOU ODETTE, Mads Christian Hansen, Marek

Kwiatkowski, Mark Donoghoe, Markus P., Daniel Edward Marthaler, Matthieu LEROY, Mattia Arsendi, Matěj, Maurits van der Meer, Max, Michael Colaresi, Michael DeJesus, Michael DeWitt, Michael Dillon, Michael Lerner, Mick Cooney, MisterMentat , Márton Vaitkus, N Sanders, Nathaniel Burbank, Nicholas Cowie, Nick S, Octavio Medina, Ole Rogeberg, Olivier Ma, Paolo, Pat LS, Patrick Kelley, Patrick Boehnke, Pau Pereira Batlle, Pieter van den Berg , ptr, quasar, Ramiro Barrantes Reynolds, Raúl Peralta Lozada, Rex, Riccardo Fusaroli, Richard Nerland, Rob Davies, Robert Frost, Robert Goldman, Robert kohn, Robin Taylor, Ryan Gan, Ryan Grossman, Ryan Kelly, S Hong, Sean Wilson, Sergiy Protsiv, Seth Axen, shira, Simon Duane, Simon Lilburn, Simon Steiger, Simone, Spencer, sssz, Stefan Lorenz, Stephen Lienhard, Steve Forrest, Steve Harris, Steven Forrest, Stew Watts, Stone Chen, Stoyan Georgiev, Susan Holmes, Svilup, Tate Tunstall, Tatsuo Okubo, Teresa Ortiz, Theodore Dasher, Thomas Siegert, Thomas Vladeck, Tobychev , Tomas Capretto, Tony Wuersch, Virgile Andreani, Virginia Fisher, Vitalie Spinu, Vladimir M, VO2 Maximus Decius, Will Farr, Will Lowe, Will Wen, William A Vauter, yolhaj, yureq, and Zach A.

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